Metadata of the chapter that will be visualized online

Series Title	Understanding Complex Systems
Chapter Title	Defining Resilience Mathematically: From Attractors To Viability
Chapter SubTitle	
Copyright Year	2011
Copyright Holder	Springer-Verlag Berlin Heidelberg

Corresponding Author	Family Name	Martin
	Particle	
	Given Name	**Sophie**
	Suffix	
	Division	
	Organization	Cemagref - LISC
	Address	24 av. des Landais 63172, Aubière, France
	Email	
Author	Family Name	Deffuant
	Particle	
	Given Name	**Guillaume**
	Suffix	
	Division	
	Organization	Cemagref - LISC
	Address	24 av. des Landais 63172, Aubière, France
	Email	
Author	Family Name	Calabrese
	Particle	
	Given Name	**Justin M.**
	Suffix	
	Division	
	Organization	Helmholtz Centre for Environmental Research-UFZ
	Address	Permoserstrasse 15, 04318, Leipzig, Germany
	Division	Smithsonian Conservation Biology Institute
	Organization	National Zoological Park
	Address	1500 Remount Rd., 22630, Front Royal, VA, USA
	Email	

Abstract

The previous chapter presents different views of resilience, starting from Holling's conceptual definition of "ecological resilience": the capacity of a system to absorb "disturbance and reorganize while undergoing change so as to still retain essentially the same function, structure, identity, and feedbacks" (Walker et al. 2004). In this chapter, we focus on operational, mathematically precise definitions of resilience. In the literature, the main mathematical definitions of resilience are based on dynamical systems theory, and more specifically on attractors and attraction basins (also related to 'regime shifts' presented in the previous chapter). We present these definitions in detail, and illustrate their utility on a relatively simple rangeland management model. Furthermore, we use the rangeland example to highlight some key limitations of attractor based definitions of resilience.

Chapter 2
Defining Resilience Mathematically: From Attractors To Viability

Sophie Martin, Guillaume Deffuant, and Justin M. Calabrese

2.1 Introduction

The previous chapter presents different views of resilience, starting from Holling's conceptual definition of "ecological resilience": the capacity of a system to absorb "disturbance and reorganize while undergoing change so as to still retain essentially the same function, structure, identity, and feedbacks" (Walker et al. 2004). In this chapter, we focus on operational, mathematically precise definitions of resilience. In the literature, the main mathematical definitions of resilience are based on dynamical systems theory, and more specifically on attractors and attraction basins (also related to 'regime shifts' presented in the previous chapter). We present these definitions in detail, and illustrate their utility on a relatively simple rangeland management model. Furthermore, we use the rangeland example to highlight some key limitations of attractor based definitions of resilience.

We then introduce the viability based definition (Martin 2004). In this approach, the desired property (function or service) of the system is defined as a given subset of the state space (containing attractors or not). This is more general than the attractor based definition in which the desired property is defined as a subset of attractors.

We underline that changing this definition of the desired properties of the system changes the mathematical framework for defining resilience, and that viability theory is the appropriate one (Aubin 1991). Indeed, viability theory focuses on

S. Martin (✉) · G. Deffuant
AQ1 Cemagref - LISC, 24 av. des Landais 63172 Aubière, France
e-mail: sophie.martin@cemagref.fr; guillaume.deffuant@cemagref.fr

J.M. Calabrese
Helmholtz Centre for Environmental Research-UFZ, Permoserstrasse 15, 04318 Leipzig, Germany

Smithsonian Conservation Biology Institute, National Zoological Park, 1500 Remount Rd., Front Royal, VA 22630 USA
e-mail: CalabreseJ@si.edu

G. Deffuant and G.N. Gilbert (eds.), *Viability and Resilience of Complex Systems*, Understanding Complex Systems, DOI 10.1007/978-3-642-20423-4_2,

dynamical systems which cease to function or badly deteriorate when they cross the limits of a subset of their state space, called the viability constraint set. Hence the problem addressed is to keep the system within the limits of this viability constraint set. The problem is formally the same, when the system should remain in a desired set. The concepts and tools from viability theory can be used directly. Viability theory methods and tools have recently been used to address some of the issues encountered in natural resource management (Béné et al. 2000, Mullon et al. 2004, Doyen et al. 2007).

We use two main concepts derived from viability theory: "viability kernel" and "capture basin". Within this framework, one can define the resilience basin as the *capture basin of the viability kernel*. We present these concepts in more detail, and illustrate them on an example of savanna simplified dynamics (Anderies et al. 2002). We show that this viability based definition of resilience includes the attractor based definition as a particular case. Moreover, it offers new possibilities:

- The approach allows one to compute policies of action to keep or restore the desired property of the system
- The desired property can be defined as a subset of the state space which does not include any attractor of the dynamical system. In this case, to keep the desired property, one should act regularly on the system. This corresponds to usual situations of ecological or social systems, which are impossible to address in the attractor based framework of resilience

2.2 Attractor Based Definition of Resilience

In this section, we summarise the main mathematical definitions of resilience based on attractors and attraction basins.

2.2.1 Main Hypothesis: The Dynamics Includes "Good" and "Bad" Attractors

First, we quickly introduce the notions of dynamical systems and attractors (for mathematical details, see for instance Murray et al. 1994).

A dynamical system is defined by several state variables $x \in \mathbb{R}^n$, and an equation stating how these variables evolve with time. Generally, this equation gives the value of the derivative of $x \in \mathbb{R}^n$ with time as a function of x.

$$\begin{cases} x'(t) = f(x(t)), \\ x(t_0) = \quad x_0. \end{cases} \tag{2.1}$$

For the computer scientist who will discretise the time into small steps of size dt, this equation gives the value of $x(t + dt)$ as a function of $x(t)$. A first order approximation is:

$$\begin{cases} x(t + dt) = x(t) + f(x(t)).dt, \\ x(t_0) \quad = \quad x_0. \end{cases} \tag{2.2}$$

Starting from point $x_0 \in \mathbb{R}^n$, one can therefore derive a trajectory which associates with time t the value $x(t)$ of the state variables at time t.

An equilibrium point $x_0 \in R^n$ is a point where the dynamics is stopped ($f(x_0) = 0$). This equilibrium is asymptotically stable if it tends to attract the states in its vicinity. It is unstable if some points in its vicinity tend to be rejected by it. Figure 2.1 represents, for a one-dimensional state variable, a stable and an unstable equilibrium as a ball in a landscape. On the peaks, the ball can be at equilibrium, but a small change in its position leads it to fall to one side or the other. On the contrary, the ball returns to the bottom of a valley when it is slightly displaced from it.

If $x_0 \in \mathbb{R}^n$ is an equilibrium, we call attraction basin the set gathering all points $x \in \mathbb{R}^n$ such that $\lim_{t\to+\infty} ||x(t) - x_0|| = 0$. Figure 2.2 displays an illustration of three stable equilibria with their attraction basins.

More generally, an attractor is a set of states (points in the state space), invariant under the dynamics, towards which neighbouring states in a given basin of attraction asymptotically approach in the course of dynamic evolution. An attractor is defined as the smallest unit which cannot be itself decomposed into two or more attractors with distinct basins of attraction. Asymptotically stable equilibria are attractors, such as limit cycles. There are also strange attractors, but we will not discuss them in this chapter.

The attractor based definition of resilience supposes that the dynamics of the system is organized around such attractors and attraction basins, and that the system is generally close to an attractor, except immediately after strong perturbations that

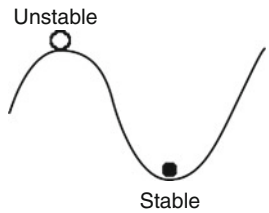

Fig. 2.1 Intuitive representation of a stable and an unstable equilibrium in a one-dimensional state space

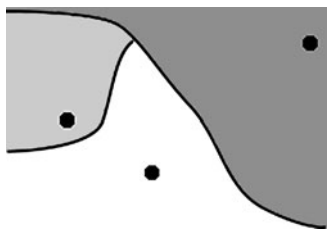

Fig. 2.2 Example of a two-dimensional state space with 3 stable equilibria and their attraction basins

can drive it far from its attractors. Moreover, it is supposed that some attractors correspond to desired (good) functioning of the system, whereas others correspond to situations that should be avoided.

2.2.2 *Illustration on Simplified Dynamics of Savanna*

To make these concepts more concrete, we illustrate them with a stylized model of rangeland management (Anderies et al. 2002). The model explores the effects of physical, ecological, and economic factors on the resilience of a rangeland system. A key problem in rangeland systems is to balance the conflicting objectives of increasing short-term profit while preserving the long-term sustainability of the rangeland. Increasing the stocking rate (animals/ha) of the rangeland increases short-term gain, but can lead to effectively irreversible degradation of the rangeland if the resultant grazing pressure is too high. Alternatively, keeping the stocking rate too low can lead to substantial loss of income. The goal of rangeland management then is to identify policies of action (changes in the stocking rate) that best reconcile these opposing objectives.

In the model of Anderies et al. (2002), good stable equilibria correspond to high shoot biomass and the bad ones to null level of shoot biomass (Fig. 2.3).

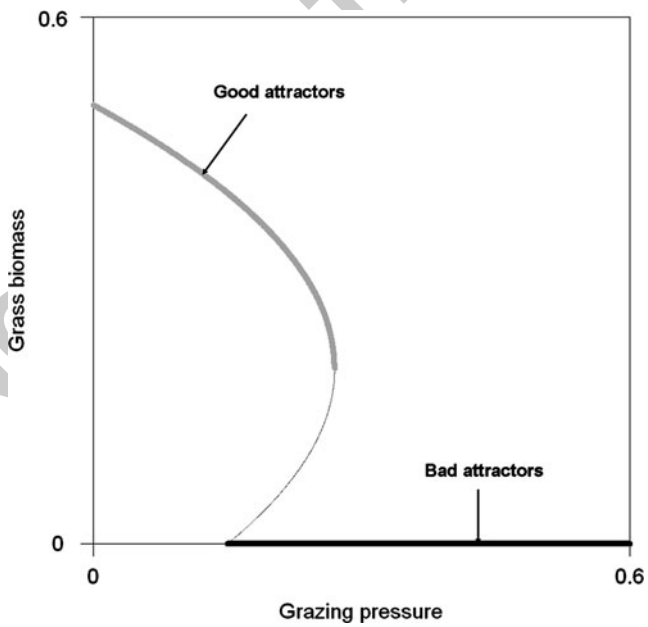

Fig. 2.3 Equilibria of the rangeland system model (2.3). Stable equilibria (attractors) are drawn with *plain lines* and unstable ones are drawn with *dashed lines*. The *arrows* represent the direction of the dynamics. 'Good' attractors (high shoot biomass) are coloured *grey* and 'bad' ones (null shoot biomass) are coloured *black*

After some simplifications in order to make the presentation easier, the evolution of the shoot biomass (grass) can be considered to be governed by the differential equation:

$$s'(t) = s(\alpha_1 + \alpha_2 s - \alpha_3 s^2) - \gamma_g s := f(s), \tag{2.3}$$

where α_1, $\alpha 2$ and α_3 are parameters, and γ_g is the grazing pressure. The first term expresses the nonlinear growth of the grass (which is actually coupled with the growth of bushes but we simplified), and the second term expresses the effect of grazing on the grass biomass. The equilibria are given by:

$$\begin{cases} s \geq 0 \\ s = 0 \\ \alpha_1 - \gamma_g + \alpha_2 s - \alpha_3 s^2 = 0. \end{cases} \tag{2.4}$$

Taking the Anderies et al. values for the parameters, we obtain the sets of stable and unstable equilibria plotted in Fig. 2.3. Notice that $s = 0$ is unstable for grazing pressure below 0.15 and stable for grazing pressure strictly greater than 0.15. The equilibria defined by

$$0.15 - \gamma_g + 1.2s - 3s^2 = 0 \tag{2.5}$$

are stable for $s > 0.2$ and unstable for $s \in [0; 0.2]$.

The first step to evaluate resilience consists in distinguishing good from bad attractors according to the desired system state. In the rangeland model, "good" attractors are high shoot biomass stable equilibria and "bad" ones are null shoot biomass stable equilibria (see Fig. 2.3).

2.2.3 Attractor Based Measures of Resilience

As far as real dynamical system models are concerned, the stability analysis is used to characterize the system response to small perturbation qualitatively: does the system return to its original state after a small perturbation, or not?

Resilience is then considered as a quantitative additional characterization either linked to the return time or the attraction basin size in the state space of the "good" asymptotically stable equilibria.

2.2.3.1 Resilience As the Inverse of Return Time

The return time is often used as a resilience index in the literature. As far as differential equation models are concerned, Pimm and Lawton (1977) and

DeAngelis (1980) for instance, used the eigenvalue with the maximal real part of the linearization to evaluate the resilience as the asymptotic rate of convergence. Actually, the asymptotic rate of convergence equals the opposite of the eigenvalue with the maximal real part (see for instance Murray et al. (1994)), and the bigger the rate of convergence is the more resilient the system is, since its state goes back more quickly to a neighbourhood of the equilibrium. To complement this index, which refers to the properties of asymptotic dynamics, Neubert and Caswell (1997) proposed indices allowing the characterization of transient responses following perturbation.

In the case of individual-based models and cellular automata, resilience is studied with simulations as the inverse of the time needed after some kind of disturbance to return to its original state (Ortiz and Wolff 2002) or to reach a certain percentage of the previous abundance (Matsinos and Troumbis 2002).

Keeping to the example of the rangeland system model of Anderies et al. (2002), the dynamics in the neighbourhood of good attractor s^* can be linearly approximated:

$$f(s^* + \epsilon) \approx \frac{df}{ds}(s^*).\epsilon \tag{2.6}$$

$$\frac{df}{ds}(s) = \alpha_1 + 2\alpha_2 s - 3\alpha_3 s^2 - \gamma_g. \tag{2.7}$$

Taking the Anderies et al. values for the parameters, we obtain:

$$\frac{df}{ds}(s) = 0.15 + 2.4s - 9s^2 - \gamma_g. \tag{2.8}$$

Considering a good attractor $s^* > 0.2$, defined by (2.5) according to the grazing pressure γ_g, the value of the linearized dynamics at s^* equals :

$$\frac{df}{ds}(s^*) = s^*(1.2 - 6s^*). \tag{2.9}$$

The resilience value, $R(s^*)$, of the system at equilibrium s^* is then the opposite of $\frac{df}{ds}(s^*)$.

Actually, $s^* = 0.2 + \sqrt{0.09 - \frac{\gamma_g}{3}}$ so s^* is defined for $\gamma_g \in [0; 0.27]$. When $\gamma_g \leq 0.27$,

$$R(\gamma_g) = -s^*(1.2 - 6s^*). \tag{2.10}$$

When $\gamma_g >= 0.27$, the resilience $R(\gamma_g)$ is null as there is no good asymptotically stable equilibrium associated with such grazing pressure intensity.

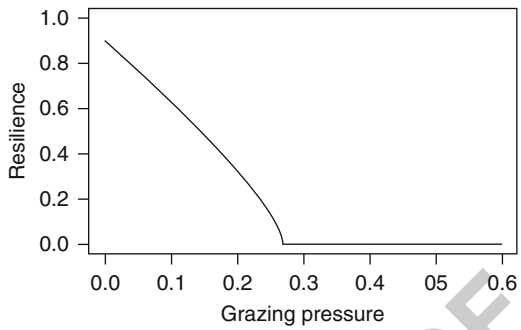

Fig. 2.4 Resilience value according to the grazing pressure γ_g, when defined as the asymptotic rate of convergence to a good attractor

Figure 2.4 shows the resilience value according to the grazing pressure γ_g. It is worth noting that this resilience measure refers to asymptotic dynamical properties, and is represented by the inverse of the asymptotic rate of convergence.

2.2.3.2 Resilience Proportional to the Attraction Basin Size

According to Beddington et al. (1976), resilience may be interpreted as the perturbation magnitude a system can absorb without experiencing qualitative changes. From a dynamical system viewpoint, "qualitative changes" may be considered as a state jump into another attraction basin. The resilience of the system is then defined as proportional to the distance between a good attractor and the boundary of its attraction basin (see for instance Collings and Wollkind 1990 and van Coller 1997).

In Anderies (2002), this measure is used to study resilience of the rangeland model. To define a resilience measure based on the distance to the attraction basin boundary, we have to compare this distance with the distance to the definition domain (Fig. 2.5).

Variable s represents the shoot (grass) biomass and is therefore positive. When a good equilibrium exists, resilience is then defined as the quotient of the distance to the boundary of the attraction basin by the difference between this distance and the distance to the definition domain :

$$R(s^*) = \frac{\min\left(s^*, 2\sqrt{0.09 - \frac{\gamma_g}{3}}\right)}{s_{ge} - \min\left(s^*, 2\sqrt{0.09 - \frac{\gamma_g}{3}}\right)} \tag{2.11}$$

Resilience is then infinite when the definition domain is included in the attraction basin of good attractors. When it is included in the attraction basin of bad attractors ($\gamma_g > 0.27$), resilience is null. Figure 2.6 displays the resilience plot.

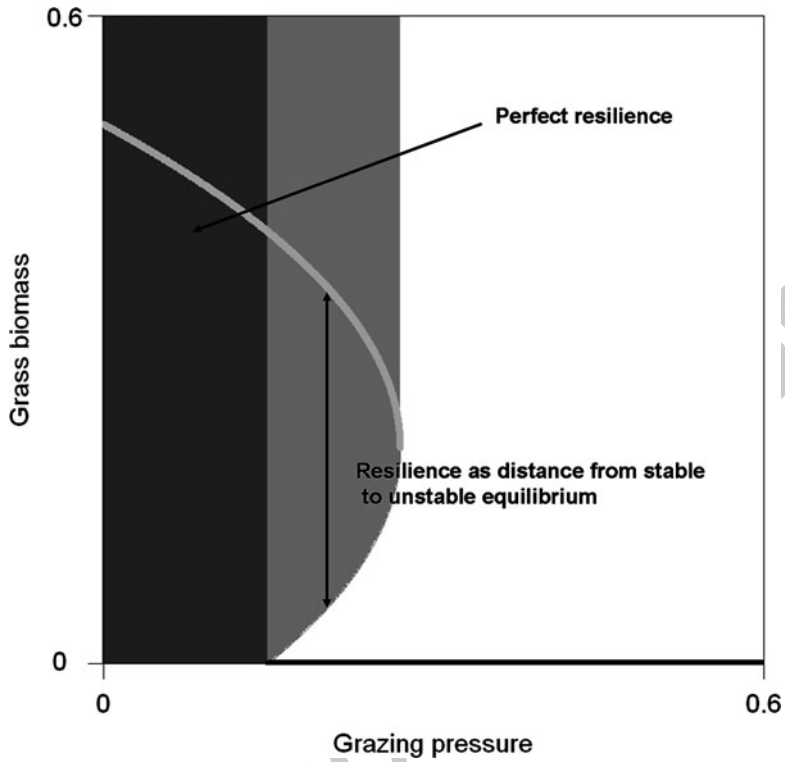

Fig. 2.5 Illustration of the distance between the good attractors and the boundary of their attraction basin

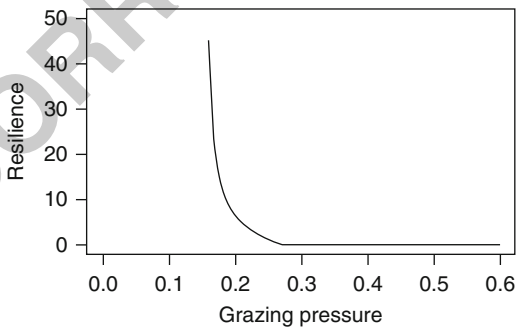

Fig. 2.6 Resilience measure based on the distance of the good attractors to the boundary of their attraction basin

It is worth noticing that such a definition of resilience is globally associated with an equilibrium. It does not distinguish between the states inside the attraction basin, whereas one could expect to get different resilience values for states near the attraction basin boundary and those which are closer to the equilibrium.

2.2.4 Limitations

2.2.4.1 The Choice of Actions is Not Included

These formalizations of resilience suppose that the feedback law or the management policy is defined, and the results only concern this particular management policy. Computing these resilience indices for different management policies allows one to compare them, but the set of possible policies is generally impossible to explore sufficiently to guarantee that a good policy is found. One needs another mathematical framework to include the choice of actions in the definition of resilience.

2.2.4.2 The Property of Interest Might not be Found in the Attractors

Both definitions suppose that the system shows the properties of interest when it reaches a given set of attractors (the good attractors). This corresponds to a widespread view in ecology that the desirable state of the system is an attractor where the system naturally goes without human intervention. But one can also often meet situations where human action and ecological dynamics are strongly interrelated. In this case, the property which we want to evaluate the resilience of is not necessarily found in an attractor.

Furthermore, inside a particular attraction basin, the resilience index should depend on the distance from the state of the system to the attraction basin boundary: resilience may be smaller near the boundary than in the middle of the attraction basin.

2.3 Viability Based Definition of Resilience: Dynamics without Management

In this section, we consider the definition of resilience based on the viability approach (Martin 2004) without including management actions. This will allow us to show that this definition can, in some conditions, coincide with the definition presented in the previous section. We shall demonstrate later that the viability based definition of resilience is more general, and can overcome some of the limitations of the attractor based definition.

2.3.1 Resilience of a Property Defined by a Subset of the State Space

In the attractor based definition of resilience, the desired property of the system is defined by a set of "good" equilibria or attractors. The main novelty of the viability

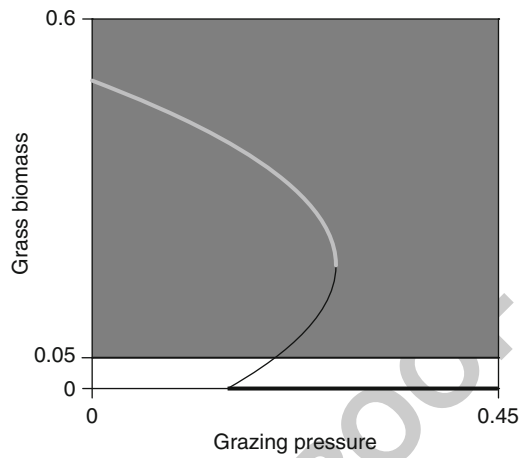

Fig. 2.7 The desired set, in *dark grey*, is defined as a level of grass higher than a threshold

based approach is to define the desired property of the system as a set of states, which are not necessarily attractors. This means that the desired property can be defined without knowing the dynamics of the system (its attractors for instance), but simply according to the desirable use of the system. For instance, in our simple rangeland model, a natural desired property of the system would be to have a grass biomass higher than a threshold (avoiding the complete disappearance of grass which would prevent the possibility of grazing). Therefore, the desired set can be, for example, as in Fig. 2.7.

This additional freedom in defining the desired property of the system has major consequences. First, we consider the resilience of the desired property rather than the resilience of the system itself. This is an important shift which requires us to be more precise. Moreover, since the considered states defining the desired property are not necessarily attractors, it is not possible any more to base the definition of resilience on attractor basins or convergence rates to the attractors. One needs a different theoretical framework, which focuses on the trajectories that remain in the desired set, or trajectories that come back to the desired set and remain in it. Fortunately, this theoretical framework exists: it is viability theory.

2.3.1.1 Viability Kernel

Viability theory was developed in the 1990s (Aubin 1991). Originally, its purpose was to study systems which collapse or badly deteriorate if they leave a given subset of the state space, called the viability constraint set. Therefore, the objective is to keep the system in the part of the state space where it can survive, i.e. where it is viable. The theory includes the case when one can act on the system to modify its trajectory, but for the moment, we focus on the particular case where no management is included (2.1), to facilitate comparison with the attractor based

definition (we shall consider the case including management actions in the next section).

In this case, a viability problem is to identify the trajectories of a dynamical system which remain in the viability constraint set, and distinguish them from the trajectories that cross its boundaries. We recognise strong similarities with the resilience issue when the desired property of the system is defined as a set of states. The only difference is that the desired property of the system is defined by its usage, and can be very different from its viability. But these are simply different interpretations of the same mathematical concepts. In viability theory, the set of trajectories which remain in the constraint set is called the *viability kernel*. More formally, if we denote $S_f(x,t) \in \mathbb{R}^n$ the point reached at time t for a trajectory starting at point $x \in \mathbb{R}^n$ at time 0, then the viability kernel $Viab_f(K) \subset \mathbb{R}^n$ of desired set $K \subset \mathbb{R}^n$ for dynamics defined by (2.2) is defined by the following equation:

$$Viab_f(K) := \{x \in K \text{ such that } \forall t \geq 0, S_f(x,t) \in K\} \tag{2.12}$$

In the rangeland model example, with the desired set K defined previously, we can compute the associated viability kernel using general methods that are presented in more detail in Chap. 7 (Deffuant et al. 2007). The result is shown in Fig. 2.8. All the points leading to grass extinction are excluded, because they cross the threshold of grass biomass. Only the points of the initial desired set leading to the good attractors are kept in the viability kernel. Thus the viability kernel is the intersection between K and the attraction basin of the "good" attractors.

2.3.1.2 Resilience Basins

In a resilience problem, we want to address also the case where a perturbation leads the system to lose the desired property, and how the system can recover it. Hence

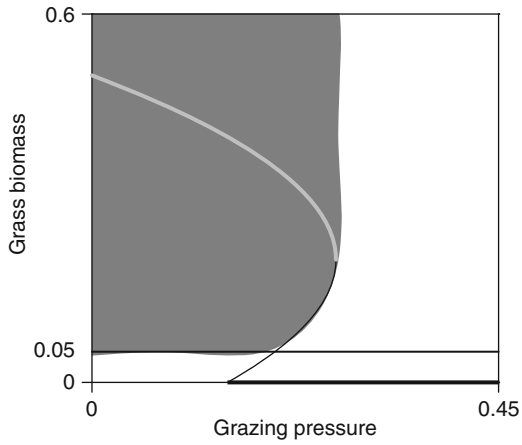

Fig. 2.8 The viability kernel in *dark grey* includes only the points from which the trajectory never crosses the limit of the constraint set

we need to get the states located outside K which go to K and remain in K. By definition the points which remain in K belong to the viability kernel, therefore, we need the points outside the kernel that go to the kernel. Actually, this problem to reach a given set as target is also addressed by viability theory (it can be seen as a particular case of the viability problem). The set of points going to a target set is called the *capture basin* of this set. More formally, if we consider a target set C, the capture basin $Capt_f(C)$ of C through dynamics defined by function f is given by:

$$Capt_f(C) = \{x \in \mathbb{R}^n \text{ such that } \exists T > 0 \text{ with } S_f(x, T) \in C\}. \tag{2.13}$$

In this framework, *the capture basin of the viability kernel defines the set of resilient states*. Indeed these are the states which go to the viability kernel of the desired property, and by definition, they remain there. In this framework, the viability kernel is analogous to the attractor in the usual framework, and the capture basin is analogous to the attraction basin. The set of resilient states is called the *resilience basin*.

Moreover, one can associate a resilience value with a point of the space as the inverse of the time to get to the viability kernel. The states belonging to the viability kernel have an infinite resilience because the time to reach the viability kernel is null. The states from which the viability kernel can be reached in finite time, have a finite strictly positive resilience value: they are resilient. Non-resilient states have a null resilience value: the time to reach the viability kernel is infinite. More generally, it is possible to define a cost for the restoration (return to the viability kernel) in this framework (Martin 2004). However, to simplify, in the following we shall suppose that the cost is only counted in time.

It is also convenient to define *finite time resilience basins*, which correspond to the states of the space which can be driven back to the viability kernel in a time which is lower than a given finite value. We shall see that practically we need to compute a set of such finite time resilience basins to derive the action policies to drive back the system into its viability kernel (see Chap. 7). In the following, we may refer simply to resilience basins, the plural implying that they correspond to finite time. Using the singular 'resilience basin' generally refers to the infinite time resilience basin.

Again, we can illustrate these concepts on our example of simplified savanna dynamics. Using the same general tools as before, one can compute resilience basins, which are represented by lines of different grey levels at the bottom of Fig. 2.9. Note that the return time gets higher and higher when the points get closer to the unstable equilibrium (boundary between attraction basins). It even reaches the limits of the computational scheme in the vicinity of the boundary between the attraction basins because there is a small strip which is considered as non-resilient, whereas theoretically, it is resilient. The dynamics is very slow in this part of the space, and the necessary time to leave it is approximated as being infinite.

If we compare this result with the attractor based definition of resilience, we note that most of the states which are resilient in the attractor based definition are viable in the viability based definition. The reason is that in the viability based

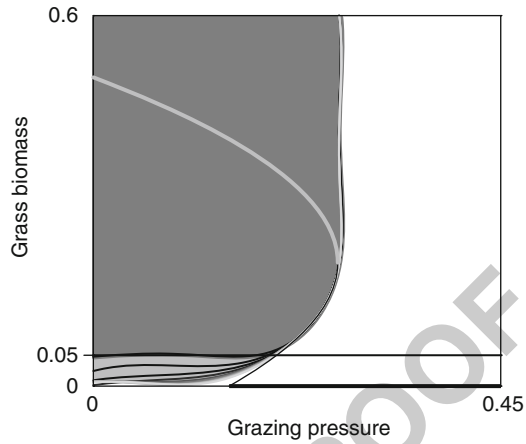

Fig. 2.9 The resilience basins include the viability kernel and some states below the constraint. The *lines* represent the resilience basins limits, and they get lighter when the resilience decreases. The *white area* is outside the resilience basins and corresponds to a null resilience (impossible to come back to the viability kernel)

definition, we only require the grass biomass to be higher than a threshold. All states satisfying this constraint are considered equivalent. Does it mean that the definitions are fundamentally different? Actually no, it depends on the definition of the desired state, as shown in the next section.

2.3.2 *Attractor Based Resilience as Particular Case of Viability Based Resilience*

In the attractor based resilience the desired set is the set of good attractors: it is assumed that the system shows the desired properties when it reaches one of these good attractors. Hence, it is interesting to test the viability based approach in the case where the desired set is the set of good attractors.

Actually, because of computational constraints for computing viability kernels, it is difficult to consider a desired set with a null surface. We therefore defined a set that surrounds the set of good attractors, as shown in Fig. 2.10.

The viability kernel of K gathers all states in K from which there exists an evolution that remains in K. Consequently, the viability kernel of K for the dynamics described by (2.3), is the intersection between K and the attraction basin (Fig. 2.11).

Having determined the viability kernel of the constraint set, we address the issue of determining whether points lying outside this viability kernel can enter it. The light grey part of the graph in Fig. 2.12 represents the infinite time resilience basin and the level curves represent growing time resilience basins (lighter curves indicate higher time).

Notice that in this case, when K is defined as a neighbourhood of the good attractors, the resilience basin and the attraction basin coincide. Therefore, both approaches lead to similar conclusions; the resilient states are the same. When

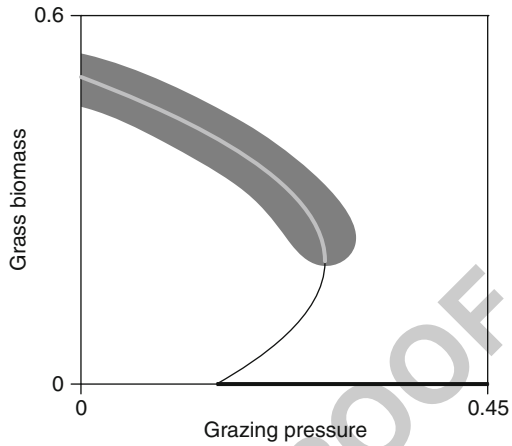

Fig. 2.10 The constraint set, K, coloured *dark grey*, is a neighbourhood of the good attractors

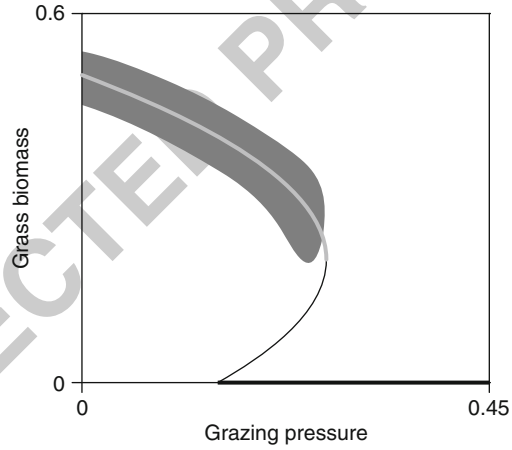

Fig. 2.11 The viability kernel of K is coloured *dark grey*. Only a small part of the constraint set is removed

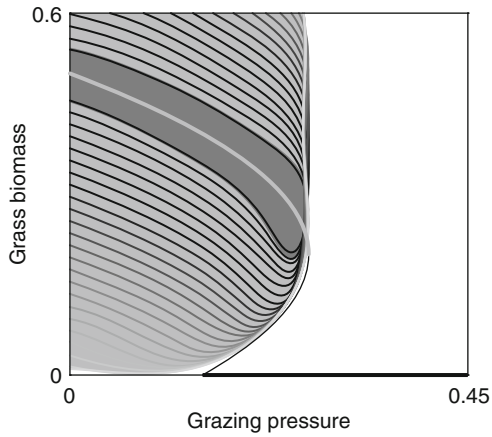

Fig. 2.12 The *dark grey* area is the viability kernel of K which coincides with the infinite resilience. The *lines* represent the limits of resilience basins, getting lighter when the resilience decreases. In the *white area*, the resilience is null

the desired set is defined as the vicinity of a set of attractors, the resilience basin coincides with the union of attraction basins. In this respect, the attractor based definition appears as a particular case of the viability based definition.

However, the resilience values are different. In the viability framework, the velocity of the dynamics close to the boundaries of the attraction basin is taken into account. For instance, it can be seen that the dynamics is very slow in the vicinity of the unstable equilibria, leading to a resilience tending to 0 (Fig. 2.12).

2.4 Viability Based Definition of Resilience: Including Management Actions

We suppose now that the evolution of the system also depends at each time t on a management action $u(t)$, with $u \in U \subset \mathbb{R}^p$. The differential equation governing the evolution of the system becomes:

$$\begin{cases} x'(t) = f(x(t), u(t)), \\ x(t_0) = \quad x_0. \end{cases} \tag{2.14}$$

When the time is discretised into small steps of size dt, we get the first order approximation:

$$\begin{cases} x(t + dt) = x(t) + f(x(t), u(t)).dt, \\ x(t_0) \quad = \quad x_0. \end{cases} \tag{2.15}$$

As mentioned previously, one strong limitation of attractor based resilience is that it cannot be used with such a system. Indeed, this system may include an infinity of trajectories starting from each point of the state space, corresponding to the infinity of action policies. Hence the attractor based approach can be used only if a policy of action $u(.) : t \to u(t) \in \mathbb{R}^p$ is a priori defined. We show now that the viability based resilience can be used to determine action policies that preserve or restore the desired property of the system.

2.4.1 Management Policy to Keep the Desired Property

Viability theory in its complete form addresses the problem of determining action policies for keeping a system inside a constraint set K. The viability kernel that we already presented above is a central concept of the theory to determine such a policy. When management actions are included in the system, the viability kernel (Aubin 1991) gathers all states from which there exists at least one action policy that keeps the system indefinitely inside K. More formally, if we denote $S_f(x, t, u(.))$ as the

state reached after time t, starting from state x and applying action policy $u(.)$, with the controlled dynamical system (2.14), the viability kernel is defined by (2.16).

$$Viab_{f,U}(K) := \{x \in K \text{ such that } \exists u(.) \text{ with } \forall t \geq 0 \; S_f(x,t,u(.)) \in K\} \quad (2.16)$$

For instance, in the rangeland model, we suppose that one can modify the grazing pressure, γ_g, which is the shoot biomass offtake per unit area. Previously, we considered that this pressure was constant. In practice, γ_g is almost certainly not constant. Furthermore, land managers may make decisions to modify this grazing pressure to control the rangeland system. In the simplified approach of this toy model, we suppose that the grazing pressure can be modified at each time t of a value $\gamma_g'(t)$, with

$$-0.02 \leq \gamma_g'(t) \leq 0.02.$$

Indeed, it makes sense to suppose that the grazing cannot increase or decrease indefinitely in a short time step.

The system dynamics is then described by the controlled dynamical system:

$$\begin{aligned} s'(t) &= \frac{r_c}{\gamma_c} s(a_c + r_s s)\left(1 - \frac{s}{s^*} - \alpha_{ws}\right) - \gamma_g s \\ \gamma_g'(t) &= u \in [-0.02; 0.02]. \end{aligned} \quad (2.17)$$

If we keep the desired set around the good attractors (Fig. 2.10), we get a viability kernel which is almost the same as the desired set (Fig. 2.13) – only a very small part on the right has been erased. It is logical that this viability kernel is bigger than in the case of a constant grazing, because modifying grazing offers the possibility to change the trajectories, and make them remain in K.

The viability kernel is important because it can be directly used to determine the action policies that keep the system inside the desired set. One is the "lazy"

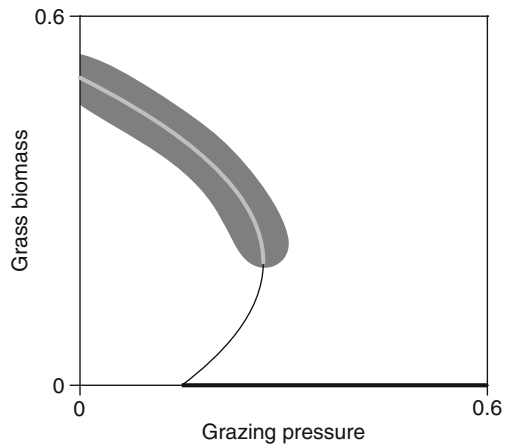

Fig. 2.13 The viability kernel of K almost coincides with the constraint set for the dynamics (2.17) including the possibility of modifying the grazing pressure with maximal rate equal to 0.02

policy, which requires that the set of actions U includes the possibility of doing nothing (action $u = 0$). Its principle is to do nothing while it is anticipated that the system does not cross the border of the viability kernel at the next time step. If it is anticipated that the system crosses this border, then is applied the first action that keeps the system inside it (one knows that such an action always exists, by definition of the viability kernel). More formally the procedure is as follows (in the case of discrete time):

- Start at point $x_0 \in Viab_{f,U}(K)$
- Denote $S_f(x_0, t, (u)_t)$ the point reached at time t, $(u)_t = (u_0, u_1, \ldots, u_{t-1})$, being the list of actions applied at each of the t time steps. Suppose $S_f(x_0, t, (u)_t) \in Viab_{f,U}(K)$. Then:
  - If $S_f(x_0, t+1, (u)_{t+1}) \in Viab_{f,U}(K)$ with $u_t = 0$, then set $u_t = 0$
  - Otherwise, set $u_t = u$ with u such that

$$S_f(x_0, t+1, (u)_{t+1}) \in Viab_{f,U}(K).$$

One can be sure that such an action u exists because $S_f(x_0, t, (u)_t) \in Viab_{f,U}(K)$. It can generally be found by a simple search, optimisation procedure or projection onto the viability kernel boundary (see Chaps. 7 and 8 for details)

2.4.2 Management Policy to Drive the System Back to the Viability Kernel

The original definition of a capture basin refers to controlled systems with management actions (2.16). The capture basin of target set C, denoted $Capt_{f,U}(C)$ is the set of states from which there exists a management policy leading the system into target set C. More formally, the definition of the capture basin is:

$$\begin{aligned} Capt_{f,U}(C) = \{x \in \mathbb{R}^n \text{ such that } \exists u(.) \\ \exists t^* > 0 \text{ with } S_f(x, t^*, u(.)) \in C\}. \end{aligned} \tag{2.18}$$

The capture basin of the viability kernel defines also the set of resilient states (the resilience basin) in the case with management actions. Indeed, this set contains the states for which there exists a management policy driving back the system into the viability kernel, where it is then possible to keep the desired property indefinitely (in the absence of perturbations). The concept is the natural extension of the one presented in the case without management actions. The concept of finite time resilience basins also extends directly.

The property of interest cannot be restored from the states outside the resilience basin, and their resilience is therefore null. The states belonging to the resilience

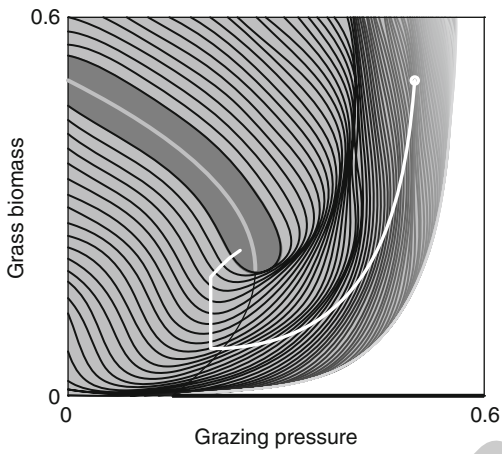

Fig. 2.14 Resilience basins for the property defined by K and dynamics described by (2.17) including the possibility of modifying the grazing pressure with maximal rate equal to 0.02. The *dark-grey area* is the viability kernel of K which coincides with the infinite resilience. The *grey lines* represent the limit of resilience basins, becoming lighter when the resilience decreases. The *white area* corresponds to a null resilience. The *white line* represents the trajectory of the system, under management actions derived from the resilience basins. It starts on the top right, outside the viability kernel, and the management actions on grazing drive back the system to the viability kernel

basin have a resilience value that is strictly positive and which may be quantified by the inverse of the time necessary to restore this property, or even with a more elaborate cost function.

As before, we can use general algorithms to apply the approach to our savanna model when management actions are considered. We note that the resilience basin is significantly larger than in Fig. 2.14, where the case without management actions was considered. This difference is expected because changing the grazing pressure provides new possibilities to drive back the system into the viability kernel.

Moreover, the computation of the set of capture basins at different time horizons (represented by the lines of different grey levels in Fig. 2.14), enables us to define an action policy that drives back the system to the viability kernel. The principle is to choose the action that makes the system go in the direction where the cost (time in the simplest case) decreases the most rapidly. In other words, this is the direction of the highest slope (considering the action possibilities) down to cost 0 which corresponds to the viability kernel. More details are provided about this in Chap. 7. Figure 2.14 gives an example of a trajectory computed using these management actions.

Therefore, we see here that the viability based definition of resilience is a generalization of the attractor based definition. It includes the attractor based definition as a particular case (when the desired set is a neighbourhood of a set of attractors and when there is no choice of action on the system), and allows us to

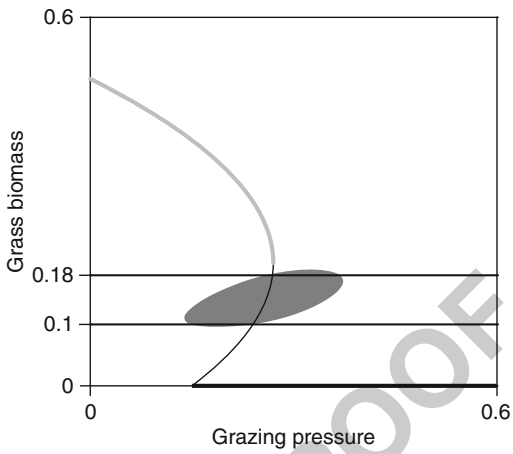

Fig. 2.15 The viability kernel of $K = [0; +\infty] \times [0.1; 0.18]$ for the dynamics (2.17) is coloured *dark grey*. K is *bordered black*

determine policies of action to keep the desired property of the system, or to recover it in the shortest possible time (or lowest cost).

2.4.3 Desired Set Without Attractor

Previously, we pointed out another limitation of the attractor based definition: the need to define the desired set as a set of attractors. The viability approach allows us to overcome this limitation. Indeed, it is possible to define a desired set without attractors, and still have a non-void viability kernel. In this case, however, it is necessary to act on the system regularly, to keep it within the desired set.

Our simplified model of savanna can illustrate this possibility. Suppose that we want to keep the level of shoot biomass between 0.1 and 0.18. This is a bit artificial, but one can imagine that the level of grass should not be too high because one wants to maximise the use of the resource. Note that this constraint set contains no attractor. We keep in the dynamics the ability of modifying the grazing pressure with absolute maximal rate 0.02 (2.17). The viability kernel is shown in Fig. 2.15.

We then compute the resilience basins, which are displayed in Fig. 2.16. The figure also shows one trajectory computed using the action policy derived from the resilience basins. As this example demonstrates, the viability framework can be applied when the viability does not include any attractor.

2.5 Conclusion and Perspectives

In this chapter, we adopted the idea that the desired property of a system can be defined as a subset of the state space without any specific conditions. This is a significant change compared with the usual mathematical approach, which supposes

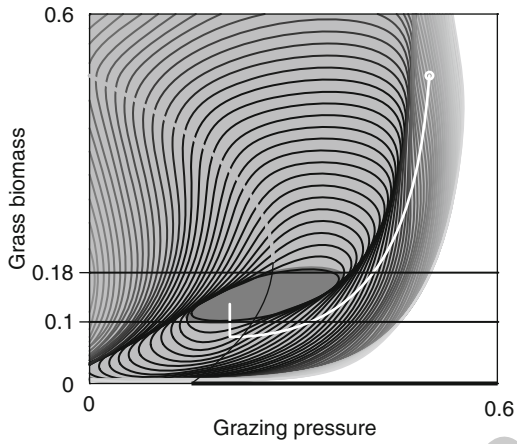

Fig. 2.16 Resilience basins for the property defined by $K = [0; +\infty] \times [0.1; 0.18]$ and dynamics described by (2.17) including the possibility of modifying the grazing pressure with maximal rate equal to 0.02. The *dark-grey area* is the viability kernel of K which coincides with the infinite resilience. The *grey lines* represent resilience basins, becoming lighter when the resilience decreases (and cost increases). In the *white area* the resilience is null. The *white line* represents the trajectory of the system under the management actions derived from the resilience basins. It starts outside the viability kernel, at the top right, and the management actions drive back the system to the viability kernel

that the desired property is defined by a set of attractors. Indeed, this leads to the adoption of viability theory as a mathematical framework for defining resilience. We can draw a parallel between attractor and viability based definitions of resilience:

- "Good" attractors are replaced by the viability kernel of the constraint set representing the desired property.
- The attraction basin of good attractors is replaced by the capture basin of the viability kernel (i.e. the points for which there exists a policy of action leading to the viability kernel).
- The resilience value as a convergence rate close to the good attractor (or as a measure of the size of the attraction basin), is replaced by the inverse of time (or more generally the cost) for driving back the system into the viability kernel.

The mathematical framework of viability theory leads to important conceptual and practical changes, compared with the attractor based framework:

- The resilience value depends on the state of the system and on its properties of interest which are defined by a subset of the state space, whereas in the attractor based framework, the resilience value is associated with an attraction basin.
- It is possible to derive action policies to keep or restore the desired property, whereas this is not the case in the attractor based framework.
- It is possible to define a desired set without any attractor, whereas this is impossible in the attractor based framework. It is worth noting that in the case

of an empty viability kernel, the capture basin of this viability kernel would also be empty, and hence there would be no resilient state, which is consistent, as no matter what the initial condition is, the desired property cannot be maintained.

We argue that the viability approach of resilience (Martin 2004) expresses better the original meaning of resilience which is to keep or restore a desired property of the system (Holling 1973). In our view, it gives a precise mathematical interpretation to this general concept, which generalises the current definitions based on attractors, without betraying the intuitive sense of the concept. One important asset of the approach is the possibility of using general algorithms to compute viability kernels, capture basins, and the associated policies of actions. However, these methods have some limitations, which are explained in more detail in Chap. 7.

The main limitation is that the computational complexity increases very rapidly with the dimensionality of the state space. Applying the approach directly on individual based models with dozens of state variables is totally excluded. Nevertheless, this method can still be used on these models, if it is possible to synthesise them into a simple dynamical system which represents adequately the main features of the dynamics.

In the next chapters of the book, we present several case studies where this global approach is applied: we consider a complex individual based model, then we synthesise it into a simplified dynamical model including a low number of state variables, and we use it to compute the viability and resilience of desirable properties.

References

Anderies JM, Janssen MA, Walker BH (2002) Grazing management, resilience, and the dynamics of a fire-driven rangeland system. Ecosystems 5:23–44

Aubin JP (1991) Viability Theory. Birkhauser, Basel

Beddington JR, Free CA, Lawton JH (1976) Concepts of stability and resilience in predator-prey models. J Anim Ecol 45:791–816

Béné D, Doyen L, Gabay D (2001) A viability analysis for a bio-economic model. Ecol Econ 36(3):385–396

Collings JB, Wollkind DJ (1990) A global analysis of a temperature-dependent model system for a mite predator-prey interaction. SIAM J Appl Math 50(5):1348–1372

DeAngelis DL (1980) Energy flow, nutrient cycling, and ecosystem resilience. Ecology 61(4): 764–771

Deffuant G, Chapel L, Martin S (2007) Approximating viability kernels with Support Vector Machines. IEEE Trans Automat Contr 52(5):933–937

Doyen L, DeLara M, Ferraris J, Pelletier D (2007) Sustainability of exploited marine ecosystems through protected areas : A viability model and a coral reef case study. Ecol Modell 208(2–4):353–366

Holling CS (1973) Resilience and stability of ecological systems. Annu Rev Ecol Syst 4:1–24

Martin S (2004) The cost of restoration as a way of defining resilience: a viability approach applied to a model of lake eutrophication. Ecol Soc 9(2):8. [online] http://www.ecologyandsociety.org/vol9/iss2/art8/

Matsinos YG, Troumbis AY (2002) Modeling competition, dispersal and effects of disturbance in the dynamics of a grassland community using a cellular automaton. Ecol Modell 149:71–83

Mullon C, Curry P, Shannon L (2004) Viability model of trophic interactions in marine ecosystems. Nat Resour Model 17:27–58

Murray RM, Sastry SS, Li Z (1994) A Mathematical Introduction to Robotic Manipulation. CRC Press, Inc., Boca Raton, FL, USA.

Neubert MG, Caswell H (1997) Alternatives to resilience for measuring the responses of ecological systems to perturbations. Ecology 78(3):653–665

Ortiz M, Wolff M (2002) Dynamical simulation of mass-balace trophic models for benthic communities of north-central Chile: assessment of resilience time under alternative management scenarios. Ecol Modell 148:277–291

Pimm SL, Lawton JH (1977) Number of trophic levels in ecological communities. Nature 268: 329–331

van Coller L (1997) Automated techniques for the qualitative analysis of ecological models : Continuous models. Conserv Ecol 1(1):5

Walker B, Holling CS, Carpenter SR, Kinzig A (2004) Resilience, adaptability and transformability in social-ecological systems. Ecol Soc 9(2): 5. [online] http://www.ecologyandsociety.org/vol9/iss2/art5/

AUTHOR QUERY

AQ1. Please check the first author considered as a corresponding author.