CHAPTER THIRTEEN

Viability analysis as an approach for assessing the resilience of agroecosystems

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Introduction

The word resilience dates back to the end of the nineteenth century, where was used in the physics of materials to describe the ability of a metal to return to its original shape after deformation produced by a shock. Its use spread over the twentieth century in the USA to fields such as: psychology, where the resilience of an individual relates to their ability to find a normal life after a trauma; to computer science, where the resilience of a system is its ability to function despite the failure of some components; and to areas of ecology, economics and social sciences. Objects change, but those that exhibit the ability to recover certain properties despite changes due to perturbations outside of their control can be described as resilient. For a physicist of materials, the system studied is a metal bar, the property its form, the disturbance – a shock, and the disruption – a deformation; for ecologists, the system might be a lake, the property may be oligotrophy (to be in a state of clear water), the disturbance – rainfall events of high intensity, and the disruption – the passage of the lake from oligotrophy to eutrophy (i.e. to be in a state of turbid water); alternatively, the ecological system might be a forest, the property – a minimum tree density to limit erosion, and the disturbance – a storm. Whatever the system, the study of resilience involves the definition of the triplet: system, properties and disturbances.

The current interest in the issue of resilience in life sciences is explained by the fact that observations of eutrophication of lakes, bleaching of corals, forests becoming savannas even after the end of cultivation, have weakened the representation of living systems as being dominated by stabilising forces which return the system to equilibrium following a disturbance. Alongside this representation is the notion that resistance is related to the intensity of the force necessary to move the system state a given distance away from the equilibrium.

In the field of ecology, the definition of resilience which enjoys a broad consensus today is the definition of Holling (1973): the ability of a system facing disturbances to maintain or recover some key properties. For example, when phosphorus discharges into a lake reach a critical level, the water becomes turbid and life (especially fish populations) is asphyxiated. In some cases, if phosphorus discharges are reduced, lake water can become clear again. In other cases, the damage is irreversible, the reduction of phosphorus inputs is not sufficient to restore clear water.

The aim of the study of resilience is to avoid situations in which natural or man-made disturbances can lead to irreversible situations. It also facilitates the restoration of the essential properties, when possible.

Holling (1996) calls this definition of resilience ‘ecosystem resilience’ or ‘ecological resilience’. Apart from its name, Levin et al. (1998) emphasise its application to the study of systems including both ecological and economic components, that is socio-ecological systems. The underlying objective is to maintain the system within certain limits rather than in a stable point.

‘Ecological resilience’ may be desirable or not: for example, a polluted resource or a dictatorship can both be very resilient. However, from the perspective of sustainability, increasing the resilience of desired properties reduces the intensity of damage caused by any disturbance.

The ability of a socio-ecological system to maintain its functioning mode depends on variables that control the boundaries of these different modes, on the intensity and the frequency of the disturbances under consideration, on the time scale (Carpenter et al., 2001), on the distribution of species or rather on the functions they perform within the same scale in time and space, and between different scales (Peterson et al., 1998).

Following the interpretation of Holling (1973), resilience depends on: (i) the state of the system, (ii) the property of interest, (iii) anticipated disturbances, (iv) dynamics and available controls, (v) the cost associated with the effort needed to restore the property of interest if it is lost following a disturbance (this cost may be the time required for recovery, for example) and (vi) the time horizon (Carpenter et al., 2001).

Several measures of resilience have been proposed to evaluate resilience in socio-ecological models.

When the model is made of differential equations, most indices are related to the eigenvalues of the linearised system near equilibrium. Pimm and Lawton (1977) measured resilience as being inversely proportional to the eigenvalue of the largest norm which represents an asymptotic property; the rate of decrease of the distance to the equilibrium when time goes to infinity. To complete the resilience index as a measure of response to small perturbations when the system is close to equilibrium, Neubert and Caswell (1997) proposed indices that assess the intensity and duration of the transient behavior near an asymptotically stable equilibrium.

According to Beddington et al. (1976), resilience can be measured as the intensity of disturbance that a system can absorb without undergoing qualitative change. From a dynamic system view, a qualitative change can be interpreted as a jump into another attraction basin. Assuming that before the disturbance, the system is in an asymptotically stable equilibrium, resilience is then defined as proportional to the distance between this equilibrium and the boundary of its attraction basin (see for example Collings & Wollkind, 1990; van Coller, 1997; Anderies et al., 2002).

Considering that resilience variations are due to slow variables, Ludwig et al. (1997) studied the parameter space and proposed the inverse of the distance to bifurcation points as a measure of resilience. Indeed, at bifurcation points, equilibria change in nature or disappear (Casagrandi & Rinaldi, 2002; Lacitignola et al., 2007).

In the case of individual-based models, resilience has been studied by simulations as the inverse of the time required for the system to recover after disturbance a state close to that before disturbance (Matsinos & Troumbis, 2002; Ortiz & Wolff, 2002).

Janssen and Carpenter (1999) studying the management of a lake showed that a choice of management can be qualified as more resilient than another when maximum values reached by the phosphorus concentration in the lake during simulation time are lower. In other words, a system is more resilient when the simulated curves are further away from dangerous areas.

In this chapter, we are interested in the issue of rangeland management using a model to describe grass dynamics (shoot biomass and crown biomass) according to the grazing pressure because an important part of rangeland management occurs through adjusting the stocking rate. It is necessary to define a control function and to specify the evolution rule for the grazing pressure as a function of time or grass biomass when using the measures of resilience described above.

Then, when the model consists of a set of differential equations, these measures of resilience give information:

- on the evolution of the system when time goes to infinity: namely on biomass and grazing levels of the attractor, the asymptotic rate of decrease of the distance to this attractor. However, transient behaviors can lead to states that are very far from the final attractor;
- on the size of the attraction basin of this attractor, but the attraction basin does not necessarily coincide with the property of minimal grazing pressure and minimal quantity of shoot biomass according to which we aim at assessing the rangeland resilience.

In the case of individual-based models, simulations can be used to evaluate the time needed for shoot biomass and grazing pressure to return to levels close to those observed before disturbance, but with no guarantee that these levels can then be preserved over time.

Thus, for the problem we address, definitions of resilience derived from the theory of dynamical systems or based on simulations suffer from three major limitations. First, they cannot be used when several policy actions are considered together, whereas the nature of socio-ecological systems is such that sets of policy actions may need to be identified in order to promote resilience. Second, these definitions limit the evaluation of the resilience of the properties of interest to attractors or unions of attraction basins, with no guarantee that these properties might actually coincide with these attraction basins. Finally, when other sets are considered, there is no guarantee that the evolution of the system will remain within these sets. However, as far as socio-ecological systems are concerned, the aim is not only to reach a satisfying state of the system but also to be able to keep this system in satisfying states over time.

The formalism of controlled dynamical systems allows us to consider from a given starting point sets of evolutions whose dynamics depend on the state of the system and also on exogenous controls which vary with time. Given prescribed properties, mathematical viability theory (Aubin, 1991; Aubin et al., 2011) develops methods and tools to determine controlled dynamical systems governing evolutions which satisfy these properties for ever or until the moment when objectives are achieved. Using this framework, which permits the design of control functions according to prescribed properties that are independent from the dynamics, Martin (2004) has proposed that the inverse of the cost associated with the effort needed to restore and preserve certain properties of the system following disturbances can be used as a measure of resilience.

Thus in a given state, for a given property and an anticipated set of disturbances:

- a system is infinitely resilient if the result of any occurrence of anticipated disturbances is that this property can be preserved;
- a system has a finite but non-zero resilience value if following an occurrence of an anticipated disturbance, this property is lost but can be restored and then preserved. Moreover, for any occurrence of anticipated disturbances, this property may be lost but can be restored and the cost associated with the restoration is bounded by the inverse of the resilience value;
- a system is not resilient if at least one anticipated disturbance causes a permanent loss of the given property (no restoration is possible).

This quantitative methodology for measuring and evaluating resilience in socio-ecological systems can be used to explore the effect of proposed policies on system resilience before they are implemented, and generates material that can help policymakers choose an appropriate policy action. In the next section, we use this viability-based measure to evaluate the resilience of rangeland management approaches that aim to preserve both a minimal shoot biomass (ecological component) and a minimal grazing pressure (economic component). We present this case study as a tutorial illustrating how viability analysis can be applied to a socio-ecological problem. We then describe the general mathematical framework used to implement the viability-based measure of resilience. We finish with some conclusions and perspectives.

Case study as a tutorial illustrating how to apply viability analysis to evaluate resilience

The measure of resilience as the inverse of the cost associated with the effort to restore and preserve certain properties of the system following disturbances involves the description of:

(1) the evolution of the system state based on the state itself, but also in terms of management actions, called controls,
(2) the properties of the system under consideration,
(3) the anticipated disturbances,
(4) the cost function used to evaluate the effort of restoration and preservation.

The dynamics, the property under study, the anticipated disturbances and the cost function

The dynamics

To describe grass dynamics, we use the model of Anderies et al. (2002) in which the grass plant is modelled as two parts, the crown and the shoots. Growth occurs through the interaction of these two parts. Incorporating the relationship between shoots and crowns yields the following model:

$$\begin{aligned} c'(t) &= r_s s(t) - c(t) \\ s'(t) &= (a_c c(t) + r_c c(t)s(t))(1 - s(t)) - \gamma_g(t)s(t) \end{aligned} \tag{13.1}$$

where c represents crown biomass, s shoot biomass, r_s, a_c and r_c are parameters that describe the rate at which crown or shoot biomass grow when crown and shoot are present. γ_g represents the grazing pressure.

In our study, grazing pressure is associated with rangeland management policies because an important part of rangeland management occurs through adjusting the stocking rate. Managers decide when to mate their ewes and rams and when to buy, sell and move their stock. However, pastoralists cannot adjust stocking rates instantaneously. Thus, we consider that the variations of the stocking rate are bounded:

$$\gamma_{g'}(t) = u(t) \in [\underline{u}; \overline{u}] \tag{13.2}$$

While Anderies et al. (2002) considered two scenarios of constant utilisation rate and constant stocking rate, we consider all policies that satisfy bounded variations of the stocking rate.

Hence we use a controlled dynamical system with a three-dimensional state space (the crown biomass, $c(t)$, the shoot biomass, $s(t)$, and the grazing pressure, $\gamma_g(t)$) and a one-dimensional control space (the variation in grazing pressure, $u(t)$):

$$\begin{aligned} c'(t) &= r_s s(t) - c(t) \\ s'(t) &= (a_c c(t) + r_c c(t)s(t))(1 - s(t)) - \gamma_g(t)s(t) \\ \gamma'_g(t) &= u(t) \in [\underline{u}; \overline{u}] \end{aligned} \tag{13.3}$$

The property under study

Our aim is to design effective policies for delivering economically and environmentally resilient agricultural systems. The property for which we are assessing the resilience has two components: on the economic side, a minimal grazing pressure, and on the ecological side, a minimal quantity of shoot biomass. This property is described by the following constraint set which is a subset of the state space (Figure 13.1):

$$\begin{aligned} \gamma_g &\geq \underline{\gamma_g} \\ s &\geq \underline{s} \end{aligned} \tag{13.4}$$

The anticipated disturbances

Next we measure the resilience of rangeland to drought events.

A period of drought causes a sudden reduction in shoot biomass. We represent a drought event as a jump in the state space from (c,s,γ_g) to $(\widetilde{c},\widetilde{s},\widetilde{\gamma}_g)$ where:

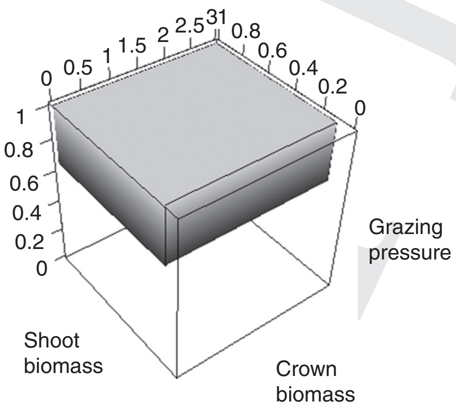

Figure 13.1 Constraint set described by (13.4) as a subset of the three-dimensional state space (c,s,γ_g) with $\underline{s} = 0.1$ and $\underline{\gamma_g} = 0.65$.

- $\widetilde{c} = c$, we assume that the drought event does not affect crown biomass,
- $\widetilde{s} = s - \alpha s$ where $\alpha \in [0, \overline{\alpha}]$ represents the severity of drought, the maximal anticipated severity is $\overline{\alpha} \quad 1$,
- $\widetilde{\gamma}_g = \gamma_g$, the drought event has no direct impact on the grazing pressure.

Thus, anticipated disturbances are jumps in the state space from (c, s, γ_g) to $(c, s - \alpha s, \gamma_g)$ where $\alpha \in [0, \overline{\alpha}]$ (Figure 13.2).

The cost function

The cost function measures the effort necessary to achieve a state from which the property under study is satisfied and can be preserved over time. In the rangeland management context, we choose to evaluate this effort in terms of time necessary to achieve a safe position, where both shoot biomass and grazing pressure are over the minimal acceptable values and for which there are rangeland management policies that ensure that the evolution of the state remains within these minimal levels over time.

Resilience evaluation

The implementation of the measurement of resilience proposed by Martin (2004) involves two steps described in the general case in section below (the viability-based measure of resilience). The results presented in this section for our case study are derived from calculations performed using the software of Patrick Saint-Pierre that implements the algorithm of Saint-Pierre (1994).

The first step consists in studying the compatibility between grass and grazing pressure dynamics described by equations (13.3), and the property of interest of the rangeland described by equation (13.4).

Actually, even if the levels of crown biomass, shoot biomass and grazing pressure are such that the property of interest is satisfied, there is no reason why this property should be maintained over time because the system state

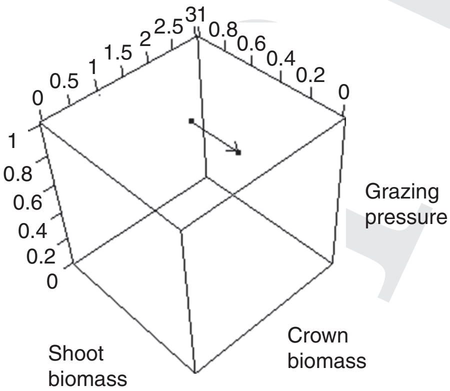

Figure 13.2 Representation of the consequence of a drought event in the three-dimensional state space. The system state jumps from the state $(c = 2, s = 0.7, \gamma_g = 0.8)$ to the state $(c = 2, s = 0.35, \gamma_g = 0.8)$. The shoot biomass has been divided by 2, the severity, α, of such a drought equals 0.5.

evolves with time and its state can exit the constraint set representing the property under study.

The viability kernel (a fundamental concept of mathematical viability theory, see below) is the subset of the constraint set gathering all the combinations of shoot biomass, crown biomass and grazing pressure that satisfy the property over time such that from an initial state belonging to the viability kernel, the rangeland can support a minimal grazing pressure while preserving a minimal quantity of shoot biomass if accurate stocking rate adjustments are fulfilled.

The size of the viability kernel depends on the possibility of stocking rate adjustments: the quicker the stocking rate variations can be performed (the bigger $|\underline{u}|$ and $|\overline{u}|$ are), the more numerous are the situations from which the property can be preserved, and consequently, the larger the viability kernel.

In the extreme case where variation in the stocking rate is not possible ($|\underline{u}| = |\overline{u}| = 0$), the grazing pressure, $\gamma_g(t)$ remains constant over time. Nevertheless, crown and shoot biomass evolves with time governed by the dynamics given in equations (13.3). Even if the initial grazing pressure and shoot biomass are over the minimal thresholds, the shoot biomass may become smaller. Given an initial grazing pressure which will remain constant, the viability kernel gathers all initial values of crown and shoot biomass such that shoot biomass remains above the threshold despite this constant grazing pressure. Clearly, as grazing pressure increases, the viability kernel will decrease in size. Figure 13.3 displays in the two-dimensional plane (c,s) viability kernels for different values of grazing pressure. Over a given value of grazing pressure, the viability kernel is empty, indicating that whatever the initial values of crown and shoot biomass, at this constant level of grazing pressure, the minimal threshold of shoot biomass is doomed to be crossed in finite time.

If the rangeland manager is now given the possibility of adjusting his stocking rate, as the crown and shoot biomass change, the third state variable, the grazing pressure may evolve with time. The viability kernel is a three-dimensional set, a subset of the constraint set displayed in Figure 13.1 and shown in Figure 13.4.

Obviously, as the maximal speed of grazing pressure variations increases, the viability kernel increases since there are more management opportunities. To illustrate this point, we show on the same graph in Figure 13.5 sections of viability kernels for $\gamma_g = 0.9$ and different values of the bounds $\underline{u}$ and $\overline{u}$. The smallest section corresponds to no grazing pressure variation and then the section surface increases as the absolute values of the bounds do.

The second step consists of evaluating the resilience of the rangeland from the impact of drought events on its ability to preserve minimal levels of shoot biomass and grazing pressure.

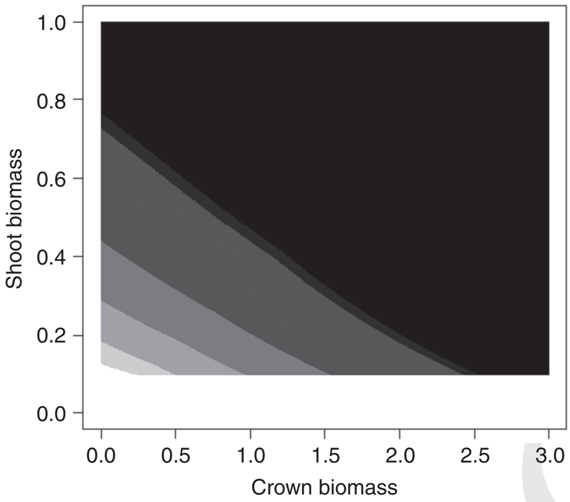

Figure 13.3 Viability kernels for dynamics (13.1) with constant grazing pressure and property (13.4) for different values of the grazing pressure. The viability kernel increases as the value of the grazing pressure decreases: for $\gamma_g = 0.905$, the viability kernel is coloured black, for $\gamma_g = 0.9$, the viability kernel extends to the darker grey area, and so on for $\gamma_g = 0.8, 0.7, 0.6$, the viability kernel extends to the lighter grey area for $\gamma_g = 0.5$. The viability kernel is empty for $\gamma_g \geq 0.91$. Parameter values are $r_s = 3$, $a_c = 0.1$, $r_c = 1$ and $\underline{s} = 0.1$.

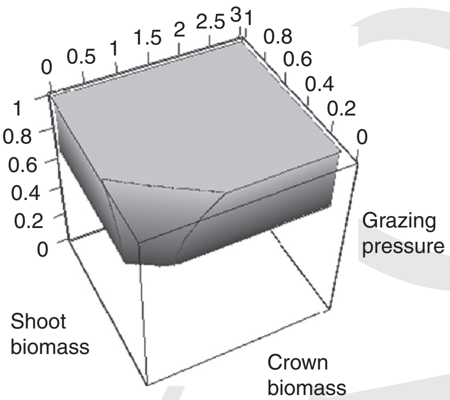

Figure 13.4 Viability kernel for dynamics (13.3) and property (13.4). Parameter values are $r_s = 3$, $a_c = 0.1$, $r_c = 1$, $\underline{\gamma_g} = 0.65$, $\underline{s} = 0.1$ and $\overline{u} = -\underline{u} = 0.05$.

Next, we measure the resilience of rangeland towards drought events. A period of drought causes a sudden reduction in shoot biomass. Thus, anticipated disturbances are jumps in the state space from (c,s,γ_g) to $(c,s-\alpha s,\gamma_g)$ where $\alpha \in [0,\overline{\alpha}]$ (Figure 13.2). Even if the state (c,s,γ_g) belongs to the viability kernel, the state $(c,s-\alpha s,\gamma_g)$ may not, implying that the property of shoot biomass and grazing pressure preservation cannot be satisfied over time whatever the stocking rate adjustments.

The damage associated with a drought event that causes a jump outside the viability kernel is measured by the time necessary to re-enter the viability kernel, if possible. This time depends on the stocking rate adjustment policy which is implemented. Thanks to viability theory tools, computing the minimal time necessary to re-enter the viability kernel enables

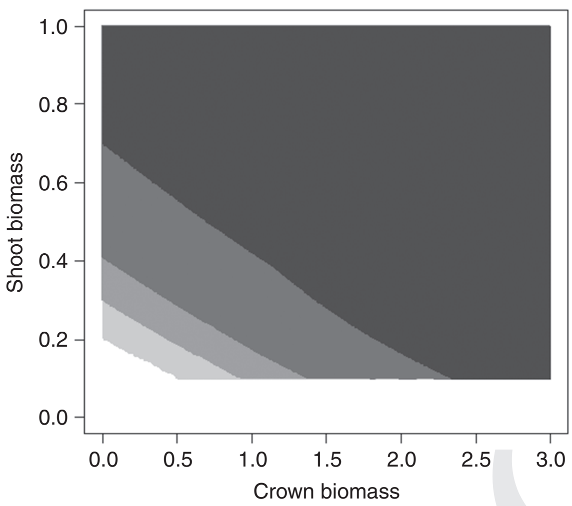

Figure 13.5 Sections of viability kernels for dynamics (13.3) and property (13.4) for $\gamma_g = 0.9$ and different values of the bounds on the grazing pressure variations. In black, the section of the viability kernel when the grazing pressure cannot be modified, $\overline{u} = - \underline{u} = 0$ (we get the same result as Figure 13.3). The viability kernel increases as the value of the bounds on the grazing pressure variations increases: for $\overline{u} = - \underline{u} = 0.03$, the section of the viability kernel extends to the darker grey area, and so on for $\overline{u} = -\underline{u} = 0.05$, the viability kernel extends to the lighter grey area for $\overline{u} = -\underline{u} = 0.1$. Other parameter values are $r_s = 3$, $a_c = 0.1$, $r_c = 1$, $\underline{\gamma_g} = 0.65$ and $\underline{s} = 0.1$.

us to derive the associated stocking rate policy (see Mathematical Viability Theory below). Following such a policy, the time spent by the rangeland system with shoot biomass or grazing pressure lower than the minimal threshold will be the lowest possible one; and the resilience will be measured from this value.

The resilience measure of the rangeland is then evaluated at any point of the state space as the inverse of the minimal restoration cost following a sudden shoot biomass reduction due to a drought event of maximal anticipated severity $\overline{\alpha}$.

Figure 13.6 displays resilience values for two different values of grazing pressure: a satisfying but relatively low grazing pressure ($\gamma_g = 0.65$) and a high one ($\gamma_g = 0.9$). From this figure we can compare the effect of level of grazing pressure on the resilience of rangeland subjected to drought events.

Figures such as Figure 13.6, which compare the resilience values associated with two values of grazing pressure, facilitate the appreciation of the impact of a decision to increase grazing pressure in terms of the loss of rangeland resilience against possible drought.

For both values of γ_g, there is a region of state space (c,s) in which the system's resilience to drought causing a halving of grass biomass is infinite. However, this surface is twice as large when the grazing pressure is $\gamma_g = 0.65$ as when it is $\gamma_g = 0.9$ (Figure 13.7: right). Similarly, in both cases, there is a space area in which the restoration time following such a drought would be more than 10 years. However, this surface is more than ten times bigger

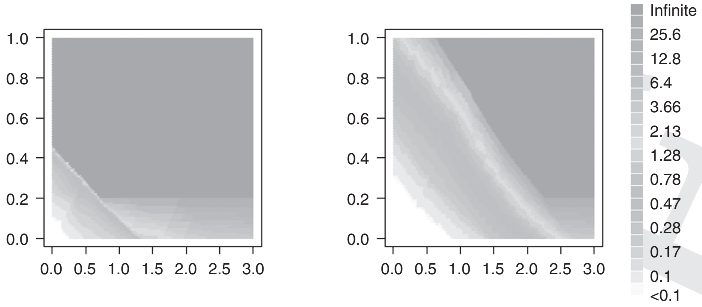

Figure 13.6 Two sections ($\gamma_g = 0.65$ (left) and $\gamma_g = 0.9$ (right)) of resilience values of the rangeland described by dynamics (13.3) for the property (13.4) toward drought events causing a sudden reduction in shoot biomass. Parameter values are $r_s = 3$, $a_c = 0.1$, $r_c = 1$, $\underline{\gamma_g} = 0.65$, $\underline{s} = 0.1$, $\overline{u} = -\underline{u} = 0.05$ and $\overline{\alpha} = 0.5$.(A black and white version of this figure will appear in some formats. For the colour version, please refer to the plate section.)

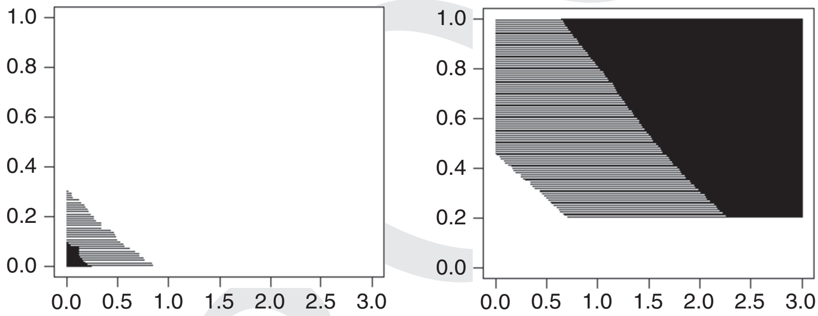

Figure 13.7 Left: the coloured black area corresponds to grass states for which resilience to drought events is smaller than 0.1 when the grazing pressure is relatively small ($\gamma_g = 0.65$). With a high grazing pressure ($\gamma_g = 0.9$), the area of resilience smaller than 0.1 increases and includes both the black and the hatched areas. Right: the coloured black area corresponds to grass states for which resilience to drought events is infinite when the grazing pressure is high ($\gamma_g = 0.9$). With a smaller grazing pressure ($\gamma_g = 0.65$), the area of infinite resilience increases and includes both the black and the hatched areas.

when grazing pressure is high $\gamma_g = 0.9$ than when it is lower $\gamma_g = 0.65$ (Figure 13.7: left).

Actually, even with grazing pressure $\gamma_g = 0.9$, the viability kernel is not empty, suggesting that from well-chosen grass states the minimal thresholds of shoot biomass and grazing pressure can be guaranteed over time; however, resilience against drought events is dramatically reduced compared to a lower grazing pressure of $\gamma_g = 0.65$.

Evolution, management and constraints are common features of socio-ecological systems. Evaluating their resilience using the viability-based measure may encourage the use of resilience thinking in environmental policy analysis. The next section focuses on the presentation of the general implementation framework.

What do we mean by viability analysis?

Mathematical viability theory

Aim and scope

Viability theory deals with the control of dynamical systems under constraints. Two reference books are Aubin (1991) and Aubin et al. (2001).

The theory has two aspects: as a mathematical theory and as a provider of mathematical metaphors of evolution of real systems.

In viability theory, the system under study is described by its state made up of a finite number of variables gathered in the finite-dimensional vector x. These variables evolve with time, such that at time t, the state of the system is described by $x(t)$.

We can distinguish direct models and inverse problems. In direct models the phenomenon under study is modelled by differential equations or rules, e.g. 'if then' in individual-based models. Once this description is completed, the study of existence, of uniqueness, of various types of stability or asymptotic stability of solutions provides answers to questions on the future behavior of the system. In an inverse problem, once the list of the prescribed properties and objectives is established, the aim is to determine dynamical systems governing evolutions that satisfy these properties for ever or until the moment when the objectives are achieved. Such questions are crucial if we no longer assume that the dynamical model for the system is well known as in Physics or Mechanics, but has to be built, as in socio-ecological systems.

Among the prescribed properties systematically studied by viability theory are properties of constraint satisfaction. For example, in the energy field, these constraints may be pollution thresholds that are not to be transgressed, or minimal requirements of energy supply. They may be satisfied at any time or until a finite, or prescribed, or minimal time when the evolution reaches a given target.

Main concepts

Given some dynamics and a constraint set, there is no reason why the evolutions governed by these dynamics should be compatible with the constraints. There are two ways of insuring viability: reduce the state space and modify the dynamics.

We consider here the first way – reducing the state space. The dynamics are given, for instance, in the form of a controlled dynamical system, the

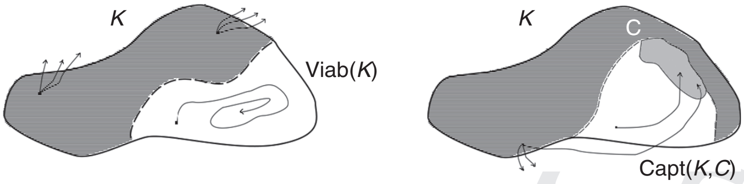

Figure 13.8 Diagrams of a set of constraints *K* and a viability kernel (left) and a capture basin (right) of target *C*.

constraint set is given as a subset of the state space, the objective is to find points inside this constraint set at which the constraints can continue to be satisfied in the future. In the terminology of viability theory, the viability kernel is the subset (possibly empty) of states within the constraint set from which at least one viable evolution (remaining all the time in this constraint set) starts. From a point inside the viability kernel there exists an evolution that remains in the constraint set. From a point outside the viability kernel, all the possible evolutions leave the constraint set in finite time (Figure 13.8: left).

If an objective is added in the form of a target to be reached inside the constraint set, viability theory uses the concept of a capture basin. This basin is the subset of states in the constraint set from which at least one evolution starts, which is viable in this constraint set until it reaches the target in finite time (Figure 13.8: right).

It is worth noting that finding the viability kernel or capture basin allows us to design feedbacks that govern evolutions so as to maintain viability until a target, if present, is captured.

Intertemporal optimality

Starting from a viability kernel or a capture basin, several evolutions might be viable. We can use the classical optimal control theories to select viable evolutions that minimise a given intertemporal criterion. Using appropriate mathematical techniques, the search for optimal evolutions becomes the search for viable evolutions for an auxiliary controlled dynamical system composed of the original system with an additional dimension which corresponds to the cost. Viability techniques can provide the same results as dynamical programming (Hamilton–Jacobi–Bellman equations), but in the presence of state constraints and for a wider class of problems (free boundary problems with obstacles).

Crisis management

Even perfect management cannot avoid crisis when some constraints are violated; for example, following a disturbance that causes a jump in the state space. Again introducing an auxiliary dynamical system, one can find control functions that allow viability to recover in the best way, for instance by minimising the crisis time (Doyen & Saint-Pierre, 1997), or, if that is not possible, by confirming the irreversibility of the constraints violation.

Exit time

Viability methods also allow us to determine for each initial position, the first time when constraints are violated. This is called the exit time function.

Available numerical tools

Few exact descriptions of viability kernels are available. Three exact descriptions of viability kernels in a two-dimensional state space are: the model called population growth in a limited space (Aubin & Saint-Pierre, 2006), the model called consumption (Aubin, 1991) and the model of Abrams Strogatz for language competition (Chapel et al., 2010). There is also the exact description of a viability domain for a three-dimensional state space (Bernard and Martin, 2012).

The possibility of exact determination of a viability kernel is studied on a case-by-case basis. The use of approximation algorithms is essential. Viability algorithms have existed since the 1990s.

The first of them by Saint-Pierre (1994) used discrete approximations in the Lipschitz case to build viability kernels. This algorithm involves two steps:

- the approximation of the viability kernel of the continuous system by kernels of discrete time systems,
- then the approximation of the viability kernels of discrete time systems by kernels of discrete systems in time and space.

Since then, other algorithms have been designed using classification procedures which are less memory-intensive than the regular grid points (Deffuant et al., 2007; Alvarez et al., 2013). Nevertheless, all these algorithms require large amounts of memory because the number of points of a regular grid increases exponentially with the dimension of the state space. Moreover, at each point of the grid, the discretised values of the set of admissible controls have to be tested; the computation time is then exponential with the dimension of the control space.

Consequently, with current calculation capacities the number of variables of the state space is in practice limited to seven.

Survey of applications

Viability theory is used in many areas: in genetics, Bonneuil and Saint-Pierre (2000) used the concept of the viability kernel to determine the initial frequencies that lead to the maintenance of polymorphism; in demography, using the tools of viability theory, Bonneuil and Saint-Pierre (2008) answered questions about lifestyle choices especially with regard to children, which can be made to guarantee a certain standard of living; in finance, the management of portfolios of financial assets can also be assessed by the tools of viability theory (Aubin, Pujal & Saint-Pierre, 2005); in aeronautics, Tomlin et al. (2003) determined the thrust and angle of attack to be applied to an aircraft to make it land under safe conditions; viability theory was also used to control food processes with the aim of identifying the set of all possible actions that make it reach a quality target with respect to manufacturing constraints (Sicard et al., 2012; Mesmoudi et al., 2014).

Considering the current forest area in the world and the current amount of CO_2 present in the atmosphere, Andrès-Domenech et al. (2011) studied the reforestation rate and CO_2 emissions required to satisfy the standards for CO_2 in the atmosphere both now and in the future. Aubin, Bernado and Saint-Pierre (2005) have evaluated the transition cost for maintaining the concentration of greenhouse gases within specified boundaries. The framework of viability theory has also been used by Béné et al. (2001) to analyse renewable resources management and by Bruckner et al. (2003) to describe the Tolerable Windows approach.

Martinet and Doyen (2007) used the tools of viability theory to define sustainability as the points at which ecological, economic and social constraints were met at the same time and for any time. Thus, each of the three pillars of sustainable development was described by a set of constraints in the state space and sustainable development was defined as the development that occurs at and remains in the intersection of these three sets of constraints.

All criteria were treated in the same way, thus avoiding the problem of having to choose weights whose values are difficult to justify as in the case of criteria based on optimisation of an aggregated utility function. Looking for sustainable developments then involves finding the conditions under which the viability kernel is not empty (as in Rapaport et al., 2006). For instance, imposing constraints that state that the levels of consumption and the stock of exhaustible resources should never fall below certain threshold limits; then in a society that is described by indexes belonging to the viability kernel, a certain level of consumption and resource conservation can be guaranteed for all generations to come. Martinet and Doyen (2007) then define sustainability as the ability to transmit a set of 'minimum rights' to future generations. Since then, several works have developed this point (De Lara & Martinet, 2009; Bernard & Martin, 2013; Wei et al., 2013; Perrot et al., 2016).

The viability-based measure of resilience

The measure of resilience proposed by Martin (2004) is the inverse of the minimal cost associated with the effort to restore and preserve some properties of the system following disturbances. Using this measure, the evaluation of resilience involves the description of:

- the evolution of the system state based on the state itself and, also on the management actions, called controls,
- the properties of the system under consideration,
- the anticipated disturbances,
- the cost function used to evaluate the effort of restoration and preservation.

The implementation in the mathematical viability theory framework

The implementation of the measurement of resilience proposed by Martin (2004) involves two stages. Given a controlled dynamical system and a property of this system, the first step is to study the system's ability to preserve this property over time. Given anticipated perturbations, the second step is to assess the impact of these disturbances on the system ability to maintain the property under consideration, which is evaluated using cost functions defined on the set of possible evolutions.

First step: the calculation of the viability kernel associated with the dynamics of the system and the desired property allows us to distinguish the states of the system from which the studied property can be preserved.

The first assumption we make is that the evolution of the system state is governed by a controlled dynamical system with changes depending on the state of the system and also on controls by an external manager.

The state of the system is described by an n-dimensional vector $x\in X \subset \mathbb{R}^n$, the controls through which an external manager can act on the system belong to a p-dimensional vector space, $u\in\mathbb{R}^p$. The controlled dynamical system S is described by the pair (U,f) where

- UU is a set-valued map $\mathbb{R}^n : \rightsquigarrow \mathbb{R}^p$ which associates any state of the system with the set of admissible controls. We use a set-valued map because the control function is not *a priori* defined. We only know that at each state of the system, several control choices are available, and all these possible choices are gathered in the set $U(x)$,
- and f is a function $\mathbb{R}^n \times \mathbb{R}^p : \mathbb{R}^n$ which associates any pair of system state and control with the variation of the system state.

An evolution $t\in[0, +\infty[x(t)\in X$ which describes the state of the system over time is governed by the controlled dynamical system S when:

$$S\begin{cases} \frac{x'(t)}{u(t)} & = f(x(t),u(t)) \\ & \in U(x(t)) \text{for almost all } t \geq 0 \ . \end{cases} \qquad (13.5)$$

It is worth noting that the control function $u(t)$ is not *a priori* defined. The only condition on $u(t)$ is that for all t $u(t)$ belongs to the set of admissible controls $U(x(t))$. Consequently, given an initial state x, there may be several evolutions satisfying (13.5) and corresponding to different control functions.

We note $S(x)$ the set of all evolutions starting at x and governed by S.

We then assume that the studied properties can be described as a subset of the state space of the system. Let K be a subset of X, we assume that the system has the properties under consideration when the state of the system belongs to K.

Thus, the system will preserve the property over time if its state follows an evolution that remains in K.

Such an evolution for which $\forall t \geq 0, x(t) \in K$ is called viable in the mathematical viability theory framework. The study of the compatibility between dynamics and properties is solved by computing the viability kernel, one of the fundamental concepts of the viability theory.

The viability kernel, $\mathrm{Viab}_S(K)$, associated with a controlled dynamical system S and subject to a constraint set $K \subset X$ gathers all states from which there exists at least one viable evolution governed by S (Aubin, 1991):

$$\mathrm{Viab}_S(K) = \{x_0 \in K \mid \exists x(.) \in S(x_0) \text{ such that} \forall t \geq 0, x(t) \in K\} \tag{13.6}$$

Second step: measuring resilience assesses the impact of disturbances on the system's ability to preserve some of its properties.

Often, when a disturbance is considered, we do not know exactly what the state of the system will be after its occurrence, we can only define a set of possible states. We then assume that the perturbations under consideration can be described by a set-valued map in the state space, D, which associates with each state of the system the set of reachable states that can occur after one of these disturbances:

$$D : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n \tag{13.7}$$

(The symbol $\rightsquigarrow$ denotes set-valued maps.)

Several disturbances can be considered such as uncertainties about the precise state of the system after their occurrence, hence the use of a set-valued map rather than a function. The anticipated disturbances are shocks in the state space.

The jump in the state space following a disturbance can be done out of the viability kernel and even outside of the set of constraints. In such cases, either the properties described by this set of constraints can be restored and preserved, that is to say that the previously calculated viability kernel can be reached by using appropriate control functions; or the properties cannot be restored, whatever the control functions.

The impact of the occurrence of a disturbance on the system's ability to preserve the properties under consideration is assessed by calculating the capture basin of the viability kernel. The capture basin is the second key concept of viability theory, which is more recent than the viability kernel. The capture basin gathers all initial states from which a target can be achieved while respecting the constraints.

The capture basin $\mathrm{Capt}_S(K,C)$ associated with the controlled dynamical system S, with the target $C \subset X$ and subject to constraints $K \subset X$ is the set of all initial points from which there exists at least one evolution viable in K until it reaches C in finite time:

$$\mathrm{Capt}_S(K,C)=\{x_0\in K|\exists x(.)\in S(x_0),\exists T\geq 0|x(T)\in C \text{ and } \forall t\in[0;T],x(t)\in K\}. \tag{13.8}$$

Thus, if after the occurrence of a disturbance the system state remains in the capture basin of the viability kernel calculated in the first step, the property can be restored and maintained. The set K which contains the set K (which represents the studied property) then represents ultimate limits beyond which the system dynamics are unknown.

From one point of the capture basin of the viability kernel, restoration is possible, but it can have a cost:

- the cost will be zero if, after the jump due to the occurrence of a disturbance, the system state remains in the viability kernel. Indeed, the property of interest will continue to be preserved,
- the cost will be non-zero but finite if the state after disturbance belongs to the capture basin of the viability kernel, because the property will necessarily be lost but can be restored and preserved,
- the cost will be infinite, if after the jump the property can not be preserved, i.e. if the system state is outside of the capture basin.

Thus, the most obvious example of cost function is the function that associates an evolution with the time spent out of the viability kernel. Let $x(.)\in S(x)$

$$c(x(.)) := \int_0^{\infty} (1 - \mathbb{1}_{Viab_S(K)}(x(\tau)))d\tau \tag{13.9}$$

where $\mathbb{1}$ denotes the indicator function of the set.

From the state $x \in X$ of the system, the minimum cost among all evolutions governed by S is:

$$c(x) = \min_{x(.)\in S(x)} \int_0^{\infty} (1 - \mathbb{1}_{Viab_S(K)}(x(\tau)))d\tau. \tag{13.10}$$

Anticipated disturbances when the system state is $x\in\mathbb{R}^n$ are described by the set $D(x)$ of reachable states after the occurrence of one of these disturbances (13.7). For the evaluation of resilience, the worst case is taken on, i.e. the jump in state space that leads to the highest cost of restoration.

The resilience of the system at state x facing disturbances described by D is equal to the inverse of the maximum cost among all jumps from x to $x_1 \in D(x)$:

$$R(x) := \frac{1}{max_{x_1 \in D(x)} c(x_1)}. \tag{13.11}$$

Scope and applications

The viability-based definition of resilience extends known measures to controlled dynamical systems and desired properties disconnected from equilibria.

Martin et al. (2011) have shown in the case of a savanna model of the literature (Anderies et al., 2002) that the definition of resilience in the context of viability theory generalises the definitions based on attractors.

In the case of measurement based on the size of the attraction basin, the resilience value is linked to the maximal intensity of the disturbance, measured in terms of sudden loss of biomass that the system at an equilibrium point can withstand while still remaining in the attraction basin of this equilibrium. The time to achieve this equilibrium is infinite.

The measure of Martin (2004) which uses a cost function associated with the return within the viability kernel of the constraint set representing the studied property is more informative. Indeed, the resilience value for any state of the system (which is not necessarily at equilibrium), gives information about the maximum time required to return the system to a given vicinity of this equilibrium following a disturbance. Moreover, this measure of resilience, together with the tools of viability theory, helps us to determine the policy actions that preserve the property of the system under consideration or, if possible, restore it within a minimal time (or minimal cost).

Viability-based measures of resilience in models of socio-ecological systems

The following studies are examples where viability analysis has been used to assess the resilience of a socio-ecological system.

In the case of lake eutrophication, Martin (2004) assessed the resilience of the oligotrophic property of a lake in the watershed of which agricultural activities were present, in the face of extreme rainfall events.

For grazing management, Martin et al. (2011) evaluated the cost associated with the restoration and preservation of a certain amount of grass biomass. Such a cost can be used to assess the resilience of the grazing system following disruptions caused by abrupt changes in livestock or drought events.

In the case of competition between languages, the evaluation of the cost associated with the restoration and the preservation of a minimum number of speakers of each language can be used to assess the resilience of the property of linguistic diversity in the face of disturbances caused by a sudden change in the proportion of speakers of each language (Bernard & Martin, 2012; Alvarez et al., 2013).

In the case of reconciling tourism and environmental quality, Wei et al. (2013) evaluated the cost associated with the restoration and preservation of situations with high standards of both tourism activity and environmental quality.

Conclusion and perspectives

The measure of resilience based on viability analysis provides an evaluation of the maximal impact of anticipated disturbances on the ability of the system to preserve or restore a given property. Hence, different agricultural systems can be compared according to their resilience values. Moreover, viability analysis can provide the management policy associated with the minimal cost of restoration.

Land management decisions need to involve all stakeholders in building the definition of the problem and its solution. This participatory management style focuses on the knowledge of each actor, discussions and negotiations. The development of methods and tools to support this participatory management process is an important area of research (Lynam et al., 2007). The modelling approach, originally proposed to help stakeholders to create a collective understanding of conflicts and to negotiate strategies for coping with them, uses multi-agent simulations to represent the evolution of environmental and economic resources and role-playing games played by stakeholders (Barreteau, 2003). Serious games are a new and efficient approach to explore and test the possibilities of changes in a realistic setting without cost or risk (Susi et al., 2007, Homewood et al., Chapter 9).

Integrating viability and resilience analysis into decision support systems for participatory management would provide stakeholders with knowledge of the resilience of the system they are currently building (Wei et al., 2012), and would enable them to examine the feasibility of using this knowledge to assess the quality of the adopted management policy.

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