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Travel Time Tubes Regulating Transportation Traffic

Jean-Pierre Aubin and Sophie Martin

This paper is dedicated to Alexander Ioffe and Simon Reich

ABSTRACT. The issue addressed is the computation at each arrival time and at each state at any node of the network, of the *travel time needed to join any node of the network to this state at this arrival date* by a prototypical vehicle and of the *regulation law* governing such viable evolutions.

We assume that is known the time dependent controlled dynamics of a prototypical vehicle, the network on which it is constrained to evolve at each instant, and the nodes from which it starts or through which it passes at prescribed times.

The basic question we address is the determination of *the arrival sets at each arrival time* made of terminal states at which arrive evolutions governed by the control system, viable in the network, starting from the departure set for *some* travel time or *prescribed* travel time. A subsidiary problem is to determine the associated nodes, through the Cournot set-valued map we define at the end of this paper.

We use viability techniques summarized in an appendix, which, translated in terms of travel time problems, allow us, for instance, to characterize the arrival tubes, define the "homoclinic" pairs at which any two nodes can be connected, prove a Lax-Hopf Formula characterizing these tubes by an easy formula whenever the control system does not depend either on the time or on the state, propose a concept of solution to a "system of Hamilton-Jacobi-Bellman inclusions" of which the arriving tube is the unique solution, adapt to those systems the dual concept of Barron-Jensen/Frankowska extension of viscosity solutions to usual Hamilton-Jacobi-Bellman solutions, optimize intertemporal criteria and minimize travel times. We derive the main properties of the Moskowitz Travel Time model of the Lighthill, Whitham and Richards' theory.

1. Introduction

This paper is motivated by some problems of traffic regulation, both for highway and aerial traffic. The problem investigated in this paper is the computation and/or optimization of travel times of a given vehicle between any points of nodes of a physical network, made of highways (Real-time traffic information on (CMS) changeable

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message signs[1] for instance) or Air Traffic Management (ATM) "Gate to Gate" or "En-Route to Enroute" 4D routes (tubes and funnels) for dynamic allocation of "available arrival routes" and computation of "4D contract flights"[2]. For highway transportation, the currently used Moskowitz/Lighthill-Whitham-Richards' theory, the main results of which are presented as examples, deserves to be extended and reformulated by addressing directly the definition and the computation of schedules.

The precise issue addressed in this paper is the computation, at each arrival time and at each state at any node of the network, of the *travel time needed to join any node of the network to this state at this arrival date*.

In this paper, for simplifying the notations, we assume that is given the control law[3] governing the evolution of the (current) state $x(t) \in X := \mathbb{R}^d$ (representing the position of the vehicle and/or its velocity) of a prototypical vehicle at any time[4].

We denote by $\mathbf{K}(t)$ a time-dependent subset describing an evolving network (nicknamed *environmental tube*). For instance, whenever the state x involves both positions and velocities, the velocity may be required to obey, for example, time dependent velocity bounds. Even in the case when subsets $\mathbf{K}(t) = K$ are constant, to say that $x \in \mathbf{K}(t)$ means that the state x lies in K at time t. Hence, the same state represented by its position $x \in K$ may belong either to $\mathbf{K}(t)$ at time t or to $\mathbf{K}(s)$ at time s at some other time s, but not to both.

The nodes of the network do form a subset $\mathbf{B}(t) \subset \mathbf{K}(t)$. This evolving subset is nicknamed the *departure tube*, even though in many examples, a viable evolution not only *starts from* $\mathbf{B}(t)$ at some departure time t, but *passes through* a node $\xi \in \mathbf{B}(t)$ at this time t (in the first case, the velocity of the vehicle is equal to zero, and in the other case, the velocity can be prescribed at this time to vary between two time-dependent bounds). As for the environmental tube, the evolving subset $\mathbf{B}(t) = B$ may be independent of t. In many examples, the subsets $\mathbf{B}(t) := \bigcup_{i \in I} \mathbf{b}_i(t)$ are discrete, and/or the subsets $\mathbf{B}(t) := \emptyset$ for all t except $t = 0,\ \tau_1,\ \cdots,\ \tau_j,\ \cdots, \tau_J$, but none of these features will be used at our level of abstraction.

In summary, we assume that the following are known:

- the time dependent controlled dynamics of a prototypical vehicle;
- the network on which it is constrained to evolve at each instant,
- and the nodes from which it *starts from* or *passes through* at prescribed times.

We now denote by $T > 0$ an *arrival time* and by $\Delta \in [0, T]$ a *travel time* (or *duration*, *age* in population dynamics, etc.). We define *arriving/travel pairs* as pairs (T, Δ) where $\Delta \in [0, T]$, with which we associate the *departure time* $T - \Delta \geq 0$.

We shall address three types of problems of increasing difficulty: characterize, study and compute:

[1] See http://www.calccit.org/projects/traveltime.html, http://www.equidyn.fr/

[2] See for instance http://www.nlr.nl/documents/AirTN/AirTNForum1-6-Derisson.pdf.

[3] even "tychastic" or "robust" in the sense of robust control against perturbations, disturbances or "tyches", an issue that will be treated in forthcoming papers.

[4] Another reason to restrict our study to the case of one prototypical vehicle is that the Viability Algorithms used to solve the problems with viability techniques require "grid algorithms" subjected to the "dimensionality curse", even though the results we present can be easily mathematically extended to any flotilla of vehicles. In this case, we suggest to use the concept of viability multipliers for modifying the dynamics of the vehicles by sending them "messages" that allow them to satisfy viability (state) constraints.

(1) *the arrival set* $\boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}(T)$ *at time* T made of terminal states $x \in \mathbf{K}(T)$ at which arrive at time T viable evolutions governed by control systems starting from $\mathbf{B}(T-\Delta)$ and viable in the tube $t \rightsquigarrow \mathbf{K}(t)$ for *some* travel time $\Delta \in [0,T]$,
(2) which is the union of *arrival/travel subsets* $\boldsymbol{A}_{(\boldsymbol{K},\boldsymbol{B})}(T,\Delta) \subset \boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}(T)$ enjoying the same properties, but for a *prescribed* travel time $\Delta \in [0,T]$,
(3) the *Cournot subset* $\mathrm{Cour}_{(\mathbf{K},\mathbf{B})}(T,\Delta,x) \subset \mathbf{B}(T-\Delta)$ made of initial states $\xi \in \mathbf{B}(T-\Delta)$ from which start viable evolutions arriving at x at time T.

We shall study successively those three maps: the arrival tube $T \rightsquigarrow \mathbf{P}_{(\mathbf{K},\mathbf{B})}(T)$ in Section 2, the arrival/travel set-valued maps $(T,\Delta) \rightsquigarrow \mathbf{A}_{(\mathbf{K},\mathbf{B})}(T,\Delta)$ in Section 5,the Cournot map $(T,\Delta,x) \rightsquigarrow \mathrm{Cour}_{(\mathbf{K},\mathbf{B})}(T,\Delta,x)$ in Section 6.

For clarity of the exposition, we detail our study in the simplest case of arrival tubes, and we only sketch more briefly analogous results for arrival/travel set-valued and Cournot maps.

We list the type or properties we obtain in this simplest framework.

We begin by providing a viability characterization of travel time tubes in Subsection 2.1. This characterization states that the graph of travel time tube is a *capture basin* (see 2.2, below) of an auxiliary target, viable in an auxiliary environment, under an auxiliary control system. It may seem strange at first glance to characterize a well-defined problem by a solution of an auxiliary and seemingly artificial capture basin problem. But this allows us to apply results surveyed and summarized in a "viability survival kit" (Section 7), which, translated in terms of travel time problems, imply without technical difficulties the properties we shall uncover. This allows us, for instance, to

- define the "homoclinic" pairs (T,Δ) at which two any nodes can be connected in Subsection 2.2, a Lax-Hopf Formula in Subsection 2.3, characterizing these tubes by an easy formula whenever the control system does not depend either on the time or on the state;
- propose a concept of solution to a "system of Hamilton-Jacobi-Bellman inclusions" of which the arriving tube is the unique solution in Subsection 2.4;
- and, in Subsection 2.5, adapt to those systems the dual concept of Barron-Jensen/Frankowska extension of viscosity solutions to usual Hamilton-Jacobi-Bellman solutions.

We investigate, in Section 3, the problem of *optimizing an intertemporal criteria* for a given arrival-travel pair (T,Δ) a viable evolution and next, minimizing the travel time. We also derive the main properties of the Moskowitz Travel Time Model in Section 4.

2. Travel Time Tubes

Definition 2.1 (Travel Time Tubes). We denote by $\mathcal{O} : (T,x) \rightsquigarrow \mathcal{O}(T,x) \in \mathcal{C}(0,+\infty;X)$ the set-valued map associating with any final time T and *arrival state* x the subset of evolutions $x(\cdot)$ governed by

$$x'(t) \;=\; f(t,x(t),u(t)) \text{ where } u(t) \;\in\; U(t,x(t)) \tag{1}$$

arriving at x at time T.

Let us consider an *environmental tube* $\mathbf{K} : t \rightsquigarrow \mathbf{K}(t) \subset X$ where $X := \mathbb{R}^d$ is a finite dimensional vector space, and a *departure tube* $\mathbf{B} : t \rightsquigarrow \mathbf{B}(t) \subset \mathbf{K}$.

We denote by $\mathcal{O}_{(\mathbf{K},\mathbf{B})} : (T,\Delta,x) \rightsquigarrow \mathcal{O}_{(\mathbf{K},\mathbf{B})}(T,\Delta,x)$ the subset of evolutions $x(\cdot) \in \mathcal{O}(T,x)$ starting from $\mathbf{B}(T-\Delta)$ at departure time $T-\Delta \in [0,T]$, viable in the environmental tube $\mathbf{K}(\cdot)$ on the interval $[T-\Delta,T]$ and arriving at x at time T.

The *travel/arrival time map* $(T,\Delta) \rightsquigarrow \mathbf{A}_{(\mathbf{K},\mathbf{B})}(T,\Delta)$ (associated with the departure tube $\mathbf{B}$ and the environmental tube $\mathbf{K}$) is the set-valued map assigning to any arriving/travel pair (T,Δ) the (possibly empty) subset $\mathbf{A}_{(\mathbf{K},\mathbf{B})}(T,\Delta)$ of elements $x \in \mathbf{K}(T)$ at which arrives at least one evolution $x(\cdot) \in \mathcal{O}_{(\mathbf{K},\mathbf{B})}(T,\Delta,x)$.

The *travel time tube* $\mathbf{P}_{(\mathbf{K},\mathbf{B})}$ is defined by

$$\mathbf{P}_{(\mathbf{K},\mathbf{B})}(T) := \bigcup_{\Delta\in[0,T]} \mathbf{A}_{(\mathbf{K},\mathbf{B})}(T,\Delta).$$

Remark. The graph $\mathrm{Graph}(\mathbf{A}_{(\mathbf{K},\mathbf{B})})$ of the arrival/travel map is nothing other than the domain of the of $\mathcal{O}_{(\mathbf{K},\mathbf{B})}$.

Examples. The choice of departure tubes allows us to cover many examples:

- If $\mathbf{B}(t) = \emptyset$ for all strictly positive time $t > 0$, then $\mathbf{A}_{(\mathbf{K},\mathbf{B})}(T) = \emptyset$ for all $\Delta < T$, so that the travel tube boils down to prescribed travel tube with travel time $\Delta = T$:
$$\mathbf{P}_{(\mathbf{K},\mathbf{B})}(T) \;=\; \mathbf{A}_{(\mathbf{K},\mathbf{B})}(T,T).$$
- If $\mathbf{B}(t) = \emptyset$ for all t except $t = \tau_0 = 0,\ \tau_1,\ \cdots,\ \tau_j,\ \cdots, \tau_J$, then
$$\mathbf{P}_{(\mathbf{K},\mathbf{B})}(T) \;=\; \bigcup_{j=0}^{J} \mathbf{A}_{(\mathbf{K},\mathbf{B})}(T,T-\tau_j).$$

Many other examples can be considered.

2.1. Viability characterization of travel time tubes. We can characterize the graph of the travel time tube with the tools of viability theory. We recall the following definition:

Definition 2.2 (Capture Basin of a Target). Let us consider a differential inclusion $x'(t) \in F(x)$ where $F : X \rightsquigarrow X$, a subset K regarded as an environment defined by (viability) constraints and a subset $C \subset K \subset X$ regarded as a target.

The subset $\mathrm{Capt}(K,C)$ of initial states $x_0 \in K$ such that C is reached in finite time before possibly leaving K by at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ of the differential inclusion starting at x_0 is called the *viable-capture basin* of C in K.

This is, together with the concept of viability kernel, one of the central concepts of viability theory, the properties of which are summarized in Section 7 below.

We shall prove the following

Proposition 2.3 (Viability Characterization of the Travel Time Tubes). *Consider the auxiliary system*

$$(2)\qquad \left\{ \begin{array}{ll} (i) & \tau'(t) = -1; \\ (ii) & x'(t) = -f(\tau(t),x(t),u(t)) \quad \text{where} \quad u(t) \in U(\tau(t),x(t)). \end{array} \right.$$

The graph of the travel time tube $\boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}(\cdot)$ is the viable-capture basin of target $\mathrm{Graph}(\boldsymbol{B})$ *viable in* $\mathrm{Graph}(\boldsymbol{K})$ *under the system (2):*

$$\mathrm{Graph}(\boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}) \;=\; \mathrm{Capt}_{(2)}(\mathrm{Graph}(\boldsymbol{K}), \mathrm{Graph}(\boldsymbol{B}))$$

Hence the graphs of the travel time tubes inherit the general properties of capture basins (see Section 7 below), that we have to translate in the framework of travel time tubes. In particular, it can be computed thanks to Viability Kernel Algorithms (see [**54, 55**], [**26**]) or level-set methods (see [**44, 45**] for instance).

PROOF. To say that (T,x) belongs to the capture basin $\mathrm{Capt}_{(2)}(\mathrm{Graph}(\mathbf{K}), \mathrm{Graph}(\mathbf{B}))$ of target $\mathrm{Graph}(\mathbf{B})$ viable in $\mathrm{Graph}(\mathbf{K})$ under auxiliary system (2) amounts to saying that there exists an evolution $(T-t, \overleftarrow{x}(t))$ governed by system (2) starting at (T,x) such that $(T-t, \overleftarrow{x}(t))$ is viable in $\mathrm{Graph}(\mathbf{K})$ until it reaches $(T-\Delta, \overleftarrow{x}(\Delta)) \in \mathrm{Graph}(\mathbf{B})$ at some travel time Δ. This means that $\overleftarrow{x}(\cdot)$ is an evolution viable in $\mathbf{K}(T-t)$ on the interval $[0,\Delta]$ and that $\overleftarrow{x}(\Delta) \in \mathbf{B}(T-\Delta)$. Setting $x(t) := \overleftarrow{x}(T-t)$ and $u(t) := \overleftarrow{u}(T-t)$, we infer that $x(T) = \overleftarrow{x}(0) = x$, $x(T-\Delta) = \overleftarrow{x}(\Delta) \in \mathbf{B}(T-\Delta)$, $x(t) \in \mathbf{K}(t)$ on the interval $[T-\Delta, T]$ and that its evolution is governed by (1):

$$x'(t) \ = \ f(t,x(t),u(t)) \text{ where } u(t) \ \in \ U(t,x(t)).$$

In other words, x belongs to $\mathbf{P}_{(\mathbf{K},\mathbf{B})}(T)$. $\square$

REMARK. In the case when $\mathbf{B}(t) = \emptyset$ for all $t < T$, we infer that

$$\mathbf{P}_{(\mathbf{K},\mathbf{B})}(0) \ = \ \mathbf{B}(0).$$

Indeed, $\mathbf{B}(0) \subset \mathbf{P}_{(\mathbf{K},\mathbf{B})}(0)$ since $\mathrm{Graph}(\mathbf{B}) \subset \mathrm{Graph}(\mathbf{P}_{(\mathbf{K},\mathbf{B})})$, on the one hand, and, on the other hand, to say that $x \in \mathbf{P}_{(\mathbf{K},\mathbf{B})}(0)$ means that the pair $(0,x) \in \mathrm{Graph}(\mathbf{P}_{(\mathbf{K},\mathbf{B})}) = \mathrm{Capt}_{(2)}(\mathrm{Graph}(\mathbf{K}), \mathrm{Graph}(\mathbf{B}))$, implying at once that $(0,x) \in \mathrm{Graph}(\mathbf{B})$, and thus, that $\mathbf{P}_{(\mathbf{K},\mathbf{B})}(0) \subset \mathbf{B}(0)$.

PROPOSITION 2.4 (Characterization of Travel Time Tubes).

(1) *The travel time tube $\boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}(.)$ is the largest* locally viable tube *between the departure tube $\boldsymbol{B}(\cdot)$ and the environmental tube $\boldsymbol{K}(\cdot)$, in the sense that for any $T > 0$, there exists $\Delta < T$ for which at least an evolution governed by (1):*

$$x'(t) \ = \ f(t,x(t),u(t)) \text{ where } u(t) \ \in \ U(t,x(t))$$

is viable in the tube $\boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}$ on $[T-\Delta, T]$.

(2) *The travel time tube $\boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}(.)$ is the smallest tube between the departure tube $\boldsymbol{B}(\cdot)$ and the environmental tube $\boldsymbol{K}(\cdot)$ satisfying the* Volterra property*:*

$$\forall\, T \geq 0, \ \ \boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}(T) \ = \ \boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})})}(T) \ := \bigcup_{\Delta \in [0,T]} \boldsymbol{A}_{(\boldsymbol{K},\boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})})}(T,\Delta)$$

which means that the travel tube associated with a departure tube $\boldsymbol{B}$ is the same as the travel tube associated with a departure tube equal to it.

(3) *The travel time tube $\boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}(.)$ is the unique tube between the departure tube $\boldsymbol{B}(\cdot)$ and the environmental tube satisfying the two properties above.*

REMARK (Volterra Property). The standard paradigm of the evolutionary system that we adopted is the initial-value (or Cauchy) problem, named after *Cauchy*. It assumes that the present time is frozen, as well as the initial state from which start evolutions governed by an evolutionary system $\mathcal{S}$.

But present time evolves, and consequences of earlier evolutions accumulate. Therefore, the questions of "gathering" present consequences of all earlier initial states arise.

There are two ways of mathematically translating this idea. The first one, the more familiar, is to take the *sum* of the number of these consequences: this leads to equations bearing the name of *Volterra*, of the form

$$\forall\, T \geq 0,\ \ x(T) \ = \ \int_0^T \theta(T-s; x(s))des.$$

The particular case is obtained for instance when the "kernel" $\theta((\cdot),(\cdot))$ is itself the flow of a determinist system $y'(t) = f(y(t))$. A solution $x(\cdot)$ to the Volterra equation, if it exists, provides at each ephemeral $T \geq 0$ the sum of the states obtained at time T from the state $x(s)$ at earlier time $T-s \in [0,T]$ of the solution by differential equation $y'(t) = f(y(t))$ starting at time 0 at a given initial state x. Then $\int_0^T \theta(T-s; x(s))ds$ denotes the sum of consequences at time T of a flow of earlier evolving initial conditions, for instance.

In the case of travel time tubes, the sum of vectors is replaced by their union, i.e., a subset.

2.2. Homoclinic travel time.

Definition 2.5 (Homoclinic Travel Time). Let us consider the environmental tube $\mathbf{K} : t \rightsquigarrow \mathbf{K}(t) \subset X$ and a departure tube $\mathbf{B} : t \rightsquigarrow \mathbf{B}(t) \subset \mathbf{K}(t)$ satisfying $\mathbf{B}(0) =: B \neq \emptyset$ and $\mathbf{B}(t) = \emptyset$ for all $t > 0$. We shall say that $T > 0$ is *homoclinic* if $B \cap \mathbf{P}_{(\mathbf{K},\mathbf{B})}(T) \neq \emptyset$. The *homoclinic time set* $\mathbb{T}$ is the subset of homoclinic times T and the *homoclinic minimum time* is $\mathbb{T}^\flat := \inf \mathbb{T}$.

Assume now that $B := \bigcup_{i \in I} B_i$ is a partition of compact subsets B_i (the pairs $B_i \cap B_j$ are empty whenever $i \neq j$). Interesting examples are provided by subsets B_i reduced to singletons $B_i := \{b_i\}$.

Since

$$\mathrm{Capt}_{(2)}(\mathrm{Graph}(\mathbf{K}), \mathrm{Graph}(\mathbf{B})) \ = \ \bigcup_{i \in I} \mathrm{Capt}_{(2)}(\mathrm{Graph}(\mathbf{K}), \mathrm{Graph}(\mathbf{B}_i)),$$

then

$$\forall\, T \geq 0,\ \ \mathbf{P}_{(\mathbf{K},\mathbf{B})}(T) \ := \ \bigcup_{i \in I} \mathbf{P}_{(\mathbf{K},\mathbf{B}_i)}(T)$$

is the union of the sub-travel time tubes $\mathbf{P}_{(\mathbf{K},\mathbf{B}_i)}(\cdot)$, where $\mathbf{P}_{(\mathbf{K},\mathbf{B}_i)}(T)$ is the set of elements $x \in \mathbf{K}(T)$ such that there exist $i \in I$ and an evolution $x(\cdot)$ governed by $x'(t) \in F(t, x(t))$, starting from B_i at departure time 0, such that $x(T) = x$ and viable in $\mathbf{K}(t)$ on the interval $[0T]$. Therefore, one can check

Lemma 2.6 (Connectivity Properties of Homoclinic Times). *Assume now that $B := \bigcup_{i \in I} B_i$ is a partition of compact subsets B_i. Therefore*

$$(3) \qquad \left\{ \begin{array}{lll} (i) & B \cap \boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}(T) \ = \ \displaystyle\bigcup_{(i,j)} B_i \cap \boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B}_j)}(T); \\ (ii) & \mathbb{T} \ = \ \displaystyle\bigcup_{(i,j)} \mathbb{T}_{(i,j)}; \\ (ii) & \mathbb{T}^\flat \ = \ \displaystyle\inf_{(i,j)} \mathbb{T}^\flat_{(i,j)}. \end{array} \right.$$

This time $\mathbb{T}^\flat_{(i,j)}$ can be regarded as the minimum travel time from B_j to $\boldsymbol{B}_i$.

If $i \neq j$, then
$$0 \;<\; \mathbb{T}^{\flat}_{(i,j)} \;\leq\; +\infty$$
because the subsets B_i are compacts, they are contained in pairwise disjoint open subsets O_i, so that the time go from one B_j to B_i is strictly positive.

For any $i \in I$, we denote by $J(T,i)$ the subset of indexes $j \in I$ such that $B_i \cap \mathbf{P}_{(\mathbf{K},\mathbf{B}_j)}(T) \neq \emptyset$. It can be regarded as a *chronosphere* around the subset B_i, denoting the family of subsets B_j from which starts at least one evolution arriving at B_i at time T. If the cardinal of $J(T,i)$ is strictly larger than 1, then T can be viewed as an *interference time* at B_i, which should be avoided for avoiding "accidents" produced by several different evolutions arriving at B_i at the same time T.

Remark (Homoclinic Pairs). If we do not assume that the tube $\mathbf{B}$ has empty values except for $t = 0$, we can extend the concept of homoclinic time. We shall say that a pair (T,Δ) made of an arrival time T and a travel time $\Delta \in [0,T]$ is *homoclinic* if $\mathbf{B}(T-\Delta) \cap \mathbf{A}_{(\mathbf{K},\mathbf{B})}(T,\Delta) \neq \emptyset$. If $\Delta = T$, then T is *homoclinic* in the sense that $\mathbf{B}(0) \cap \mathbf{A}_{(\mathbf{K},\mathbf{B})}(T,T) \neq \emptyset$. The adaptation of the above results is trivial.

2.3. The Lax-Hopf Formula. We can recover many results related to the Lax-Hopf formula for Hamilton-Jacobi-Bellman equation (see for instance [**18**], [**21**], [**51**]) from a Lax-Hopf Formula for tubes:

Theorem 2.7 (Lax-Hopf Formula for Travel Time Tubes). *Assume that $f(t,x,u) = u$ and that $u \in U$ where U is a closed convex subset and that the graph of the environmental tube $\boldsymbol{K}$ is convex. Then the Lax-Hopf Formula for tubes*

$$\forall\, T \geq 0,\ \forall\, \Delta \in [0,T],\ \ \boldsymbol{A}_{(\boldsymbol{K},\boldsymbol{B})}(T,\Delta) \;=\; \boldsymbol{K}(T) \cap (\boldsymbol{B}(T-\Delta) + \Delta U) \tag{4}$$

holds true.

Proof. Let $x \in \mathbf{A}_{(\mathbf{K},\mathbf{B})}(T,\Delta)$ and let us consider a viable evolution $x(\cdot) \in \mathcal{O}(T,x)$ governed by the control system $x'(t) \in U$ arriving at $x = x(T)$ at time T and starting from $\xi \in \mathbf{B}(T-\Delta)$ at initial time $T-\Delta$. Therefore,
$$\frac{x-\xi}{\Delta} \;=\; \frac{1}{\Delta}\int_{T-\Delta}^{T} x'(t)dt \;\in\; \frac{1}{\Delta}\int_{T-\Delta}^{T} Udt \;=\; \overline{\text{co}}(U) \;=\; U$$
so that $x \in \xi + \Delta U \subset \mathbf{B}(T-\Delta) + \Delta U$.

On the other hand, let us take $x \in \mathbf{K}(T) \cap (\mathbf{B}(T-\Delta) + \Delta U)$. Hence, there exist $u \in U$ and $\xi \in \mathbf{B}(T-\Delta)$ such that $x = \xi + \Delta u$. The evolution $x(\cdot) : t \mapsto x(t) := x + (t-(T-\Delta))u$ is a solution to differential equation $x'(t) = u \in U$ starting at $\xi \in \mathbf{B}(T-\Delta)$ and satisfying $x = x(T) = \xi + \Delta u \in \mathbf{K}(T)$.

It remains to prove that $x(\cdot)$ is viable in the environment tube $\mathbf{K}(\cdot)$. Since $\xi \in \mathbf{B}(T-\Delta) \subset \mathbf{K}(T-\Delta)$ and since $\xi + \Delta u \in \mathbf{K}(T)$, we observe that for all $t \in [T-\Delta, T]$:
$$\begin{aligned}(t,x(t)) &= \frac{T-t}{\Delta}(T-\Delta,\xi) + \left(1 - \frac{T-t}{\Delta}\right)(T,\xi+\Delta u)\\ &\in\; \frac{T-t}{\Delta}\text{Graph}(\mathbf{B}) + \left(1 - \frac{T-t}{\Delta}\right)\text{Graph}(\mathbf{K}).\end{aligned}$$

The graph of the environmental tube $\mathbf{K}$ being assumed convex, we infer that $(t,x(t)) \in \text{Graph}(\mathbf{K})$, i.e., that $x(t) \in \mathbf{K}(t)$ for all $t \in [T-\Delta, T]$. This means that the solution $t \mapsto \xi + tu$ starts at $\xi \in \mathbf{B}(T-\Delta)$, is viable in the tube $\mathcal{K}$

on the interval $[T-\Delta, T]$ and arrives at $x = x(T)$ at time T. This means that $x \in \mathbf{A}_{(\mathbf{K},\mathbf{B})}(T,\Delta)$. □

2.4. Partial differential inclusions and the regulation of viable traffic. We now characterize the travel time tube as the unique set-valued solution of a partial differential inclusion. The concept of solutions to partial differential inclusions has a rich history: see [**12, 13, 14, 16, 17**], [**25**], [**27, 28, 29**], [**7**], for instance.

Let $L \subset X$ and $M \subset X$ be two subsets of a vector space X. The subset $L - M := \bigcup_{y\in M}(L-y)$ is *Minkowski sum* of L and $-M$ and $L \ominus M := \bigcap_{y\in M}(L-y)$ is the *Minkowski difference* $L \ominus M$ between L and M.

Let us denote by

$$\widehat{D}\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t,x)(1) := -D\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t,x)(-1) \tag{5}$$

the (forward) *symmetric derivative* of the tube $\mathbf{P}_{(\mathbf{K},\mathbf{B})}$ at a point (t,x) of its graph.

Definition 2.8 (Travel Tube Partial Differential Inclusion). The travel tube is a contingent solution to partial differential inclusion if

$$0 \;\in\; \widehat{D}\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t,x)(1) - f(t,x,U(t,x)).$$

The associated *travel regulation map* $R(t,x)$ is defined by

$$R(t,x) \;:=\; \{u \in U(t,x) \text{ such that } 0 \;\in\; D\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t,x)(-1) + f(t,x,u)\}.$$

The travel time tube $\mathbf{P}_{(\mathbf{K},\mathbf{B})}(\cdot)$ is the Frankowska solution to partial differential inclusion if :

$$\left\{\begin{array}{lll} (i) & 0 \;\in\; \widehat{D}\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t,x)(1) - f(t,x,U(t,x)); \\ (ii) & 0 \;\in\; D\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t,x)(1) \ominus f(t,x,U(t,x)). \end{array}\right. \tag{6}$$

The Frankowska solution to partial differential inclusion (6), can be written in the form

$$\left\{\begin{array}{ll} (i) & \exists\, u \;\in\; U(t,x) \text{ such that } 0 \;\in\; D\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t,x)(-1) + f(t,x,u); \\ (ii) & \forall\;\, u \;\in\; U(t,x) \text{ such that } 0 \;\in\; D\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t,x)(1) - f(t,x,u). \end{array}\right.$$

Theorem 2.9 (The Travel Tube as a Frankowska Solution to a Partial Differential Inclusion). *If the control system is Marchaud, if* $\boldsymbol{B}$ *is closed and the environmental tube* $\boldsymbol{K}$ *is closed, then the travel tube* $\boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}$ *is the largest closed contingent solution to*

$$0 \;\in\; \widehat{D}\boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}(t,x)(1) - f(t,x,U(t,x))$$

satisfying $\boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}(0) = B$.

Evolutions $x(\cdot) \in \mathcal{O}(T,x)$ *viable in the travel time tube* $\boldsymbol{P}_{\mathcal{K}}(\cdot)$ *are governed by controls governing the evolutions*

$$x'(t) \;=\; f(T-t, x(t), u(t)) \;\text{ where }\; u(t) \in R(T-t, x(t))$$

starting from $\boldsymbol{B}$.

If we assume furthermore that the control system is Lipschitz, then the travel tube $\boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}$ *is the* unique *closed Frankowska solution to partial differential inclusion (6).*

Another consequence of Viability Theorems is the following property:

Proposition 2.10 (Properties of Frankowska Solutions). *We posit the assumptions of Theorem 2.9.*

For any $x \in \boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}(T)$,

- *there exists one evolution starting from* $\boldsymbol{B}$*, viable in the travel tube* $\boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}(\cdot)$ *until it arrives at time* T *at* x*,*
- *and all evolutions starting at* x *at time* T *are viable in* $\boldsymbol{P}_{(\boldsymbol{K},\boldsymbol{B})}(\cdot)$ *as long as they are viable in the environmental tube.*

2.5. Dual Frankowska solutions to PDI. The dual version of this partial differential inclusion translates the dual Frankowska property, involving the coderivative $D^\star\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t,x)$ defined by

$$p_t \ \in \ D^\star\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t,x)(p_x) \text{ if and only if } (p_t,-p_x) \ \in \ N_{\mathrm{Graph}(\mathbf{P}_{(\mathbf{K},\mathbf{B})})}(t,x).$$

Definition 2.11 (Hamiltonian of the Travel PDI). We introduce the Hamiltonian H defined by

$$\forall\, p_x \in X^\star, \ \ H(t,x,p_x) \ := \inf_{u\in U(t,x)} \langle p_x, f(t,x,u)\rangle\,.$$

The Hamiltonian $H_{\mathbf{K}}$ is defined by

$$\forall\, p_x \in X^\star, \ \ H_{\mathbf{K}}(t,x,p_x) \ := \inf_{u\in U(t,x) \mid f(t,x,u)\ \in\ D\mathbf{K}(t,x)(-1)} \langle p_x, f(t,x,u)\rangle\,.$$

In this case, the regulation map can be written in the form

$$\begin{array}{l} R(t,x) := \\ \{u\in U(t,x) \ \text{ such that } \ \forall\,(p_t,p_x)\in \mathrm{Graph}(D^\star\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t,x)), \ \langle p_x, f(t,x,u)\rangle \ \le \ p_t\}. \end{array}$$

The *dual Frankowska property* (see (27)), states that

$$\left\{\begin{array}{ll} (i) & \forall\, t>0, \ \forall\, x\in(\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t)\cap \overset{\circ}{\mathbf{K}}(t))\setminus\mathbf{B}(t), \\ & \forall\,(p_t,p_x)\in \mathrm{Graph}(D^\star\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t,x)), \ \ p_t \ = \ H(t,x,p_x); \\ (ii) & \forall\, t\ge 0, \ \forall\, x\in(\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t)\setminus \overset{\circ}{\mathbf{K}}(t))\setminus\mathbf{B}(t), \\ & \forall\,(p_t,p_x)\in \mathrm{Graph}(D^\star\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t,x)), \ \ p_t \ \in \ [H(t,x,p_x), H_{\mathbf{K}}(t,x,p_x)]; \\ (iii) & \forall\, t\ge 0, \ \forall\, x\in(\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t)\setminus \overset{\circ}{\mathbf{K}}(t))\cap\mathbf{B}(t), \\ & \forall\,(p_t,p_x)\in \mathrm{Graph}(D^\star\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t,x)), \ \ p_t \ \le \ H_{\mathbf{K}}(t,x,p_x). \end{array}\right.$$

In this case, the retroaction map R can be written

$$\begin{array}{c} \forall\, t>0, \ \forall\, x\in(\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t)\setminus \overset{\circ}{\mathbf{K}}(t))\setminus\mathbf{B}(t), \ \ R(t,x) \ := \ \{u\in U(t,x) \text{ such that} \\ \forall\,(p_t,p_x)\in \mathrm{Graph}(D^\star\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t,x)), \ \ \langle p_x, f(t,x,u)\rangle \ = \ p_t\}. \end{array}$$

Remark. To say that the concave Hamiltonian $H(t,x,p_x) := H(p_x)$ does not depend on (t,x) amounts to saying that $f(t,x,u)=u$ and that $u\in U$ where U is a closed convex subset and that $H(p_x) = \inf_{u\in U}\langle p_x,u\rangle$. The dual Frankowska characterization of travel time tubes states that $\mathbf{P}_{(\mathbf{K},\mathbf{B})}$ is the solution to the PDI

$$\begin{array}{c} \forall\, t>0, \ \forall\, x\in(\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t)\cap \overset{\circ}{\mathbf{K}}(t))\setminus\mathbf{B}(t), \\ \forall\,(p_t,p_x)\in \mathrm{Graph}(D^\star\mathbf{P}_{(\mathbf{K},\mathbf{B})}(t,x)), \ \ p_t \ = \ H(p_x). \end{array}$$

The Lax-Hopf formula for travel time tubes holds true whenever the Hamiltonian $H(t,x,p_x) := H(p_x)$ is independent of (t,x). See Section 4 below.

3. Optimal Travel Time Evolutions

We introduce an extended function $\mathbf{b}$, regarded as a (time dependent) spot cost function, with which we associate the departure tube $\mathbf{B}(t) := \{x \text{ such that } \mathbf{b}(t,x) < +\infty\}$ and $\mathbf{l} : \mathbb{R}_+ \times X \times \mathcal{U} \mapsto \mathbb{R}_+ \cup \{+\infty\}$, regarded as a transient cost function.

In this section, we keep the notations $\mathcal{O}_{(\mathbf{K},\mathbf{B})}(T,\Delta,x)$ to denote the subset of state-control evolutions $(x(\cdot),u(\cdot))$ (instead of *state evolutions* $x(\cdot)$) governed by (1):

$$x'(t) \ = \ f(t,x(t),u(t)) \text{ where } u(t) \ \in \ U(t,x(t))$$

starting from $\mathbf{B}(T-\Delta)$ at some departure time $T-\Delta \in [0,T]$, viable in the environmental tube $\mathbf{K}(\cdot)$ on the interval $[T-\Delta,T]$ and arriving at x at time T.

We are looking for travel time (duration) $\Delta \in [0,T]$ and state-control evolutions $(x^\star(\cdot),u^\star(\cdot)) \in \mathcal{O}(T,x)$ which minimize the cost function

$$J(T,x;\Delta,x(\cdot),u(\cdot)) \ := \ \mathbf{b}(T-\Delta,x(T-\Delta)) + \int_{T-\Delta}^{T} \mathbf{l}(T-t,x(t),u(t))dt$$

with respect to travel time Δ also: the function V defined by

$$\begin{aligned} & V(T,x) \\ & := \inf_{\Delta\in[0,T]} \inf_{(x(\cdot),u(\cdot))\in\mathcal{O}(T,x)} \left(\mathbf{b}(T-\Delta,x(T-\Delta)) + \int_{T-\Delta}^{T} \mathbf{l}(T-t,x(t),u(t))dt\right) \\ & = \mathbf{b}(T-\Delta,x^\star(T-\Delta)) + \int_{T-\Delta}^{T} \mathbf{l}(T-t,x^\star(t),u^\star(t))dt \end{aligned}$$

is the *valuation function* of this optimal time problem (contrary to the "value function", the time in the valuation function denotes the evolving horizon T and not the current time $t \in [0,T]$ for a fixed horizon T).

Examples. Among the many examples, we single out the following ones:

- If we assume that $\mathbf{b}(t,x) = +\infty$ for all $t > 0$, then the departure time $T-\Delta$ fixed at 0 and the optimal travel time evolution minimizes the functional

$$\begin{aligned} V(T,x) := & \inf_{(x(\cdot),u(\cdot))\in\mathcal{O}(T,x)} \left(\mathbf{b}(x(0)) + \int_0^T \mathbf{l}(T-t,x(t),u(t))dt\right) \\ = & \ \mathbf{b}(x^\star(0)) + \int_0^T \mathbf{l}(T-t,x^\star(t),\overline{u}(t))dt \end{aligned}$$

  with fixed departure time.
- If we assume that $\mathbf{b}(t,x) = +\infty$ for all t except $t = 0, \ \tau_1, \ \cdots, \ \tau_j, \ \cdots, \tau_J$, then the travel times are equal to $\Delta_j = T - \tau_j$ and the optimal travel time evolution minimizes the functional

$$V(T,x) := \min_{j=1,\ldots,J} \inf_{(x(\cdot),u(\cdot))\in\mathcal{O}(T,x)} \left(\mathbf{b}(x(\tau_j)) + \int_{\tau_j}^T \mathbf{l}(T-t,x(t),u(t))dt\right)$$

  with a finite number of fixed departure times.

The viability theorems allow us to characterize this valuation function and the retroaction map governing the evolution of optimal state-control evolutions. For that purpose, we introduce the auxiliary characteristic system

$$
(7) \qquad \begin{cases} (i) & \tau'(t) = -1; \\ (ii) & x'(t) = -f(\tau(t), x(t), u(t)); \\ (iii) & y'(t) = -\mathbf{l}(\tau(t), x(t), u(t)) \text{ where } u(t) \in U(\tau(t), x(t)) \end{cases}
$$

the auxiliary environment $\text{Graph}(\mathbf{K}) \times \mathbb{R}_+$ and the auxiliary target $\mathcal{E}p(\mathbf{b})$.

The viability theorems allow us to characterize this new valuation function and the associated retroaction governing the evolution of optimal time and state-control evolutions.

Theorem 3.1 (Viability Characterization of Optimal Travel Time Evolutions and their Regulation). *The valuation function* $V(T,x)$ *is equal to*

$$
V(T,x) \;=\; \inf_{(T,x,y)\in \text{Capt}_{(7)}(\text{Graph}(\boldsymbol{K})\times\mathbb{R}_+, \mathcal{E}p(\boldsymbol{b}))} y
$$

where $\text{Capt}_{(7)}(\text{Graph}(\boldsymbol{K}) \times \mathbb{R}_+, \mathcal{E}p(\boldsymbol{b}))$ *is the viable-capture basin of target* $\mathcal{E}p(\boldsymbol{b})$ *viable in* $\text{Graph}(\boldsymbol{K} \times \mathbb{R}_+)$ *under system (7).*

If the system is Marchaud, the function $\boldsymbol{b}$ *is lower semicontinuous and the function* $\boldsymbol{l}$ *lower semicontinuous and convex with respect to the control* u*, then the value function* V *is also lower semicontinuous.*

For characterizing the regulation map, we need to introduce the Hamiltonian $H : \mathbb{R}_+ \times X \times X^\star \mapsto \mathbb{R}$ defined by

$$
\forall\, (x,p) \in X \times X^\star,\;\; H(t,x,p) \;:=\; \inf_{u\in U(t,x)} (\langle p, f(t,x,u)\rangle + \mathbf{l}(t,x,u))
$$

with which we associate the set-valued map $M_H : \mathbb{R}_+ \times X \times X^\star \rightsquigarrow \mathcal{U}$ defined by

$$
M_H(t,x,p) \;:=\; \{u \in U(t,x) \text{ such that } \langle p, f(t,x,u)\rangle + \mathbf{l}(t,x,u) \;=\; H(t,x,p)\}.
$$

Let us consider the Hamilton-Jacobi-Bellman partial differential equation

$$
(8) \qquad \frac{\partial V(t,x)}{\partial t} \;=\; H\left(t, x, \frac{\partial V(t,x)}{\partial x}\right)
$$

satisfying the condition

$$
\forall\, t \geq 0,\; \forall\, x \in \mathbf{B}(t),\;\; V(t,x) \;\leq\; \mathbf{b}(t,x).
$$

Remark. In the case of fixed departure time obtained by assuming that $\mathbf{b}(t,x) = +\infty$ whenever $t > 0$, this condition boils down to the initial (Cauchy) condition

$$
\forall\, x \in \text{Dom}(\mathbf{b}(0,\cdot)),\;\; V(0,x) \;\leq\; \mathbf{b}(0,x)
$$

For the time-dependent Hamilton-Jacobi partial differential equation in this context, see [**46**].

Knowing the solution $V(\cdot,\cdot)$ of this Hamilton-Jacobi-Bellman equation and its derivative, we can define the regulation map driving optimal evolutions

$$
R(t,x) \;:=\; \left\{ u \in U(t,x) \text{ such that } \left\langle \frac{\partial V(t,x)}{\partial x}, f(t,x,u) \right\rangle + \mathbf{l}(t,x,u) \;\leq\; \frac{\partial V(t,x)}{\partial t} \right\}
$$

which, under additional assumption (Lipschitzianity), boils down to

$$
R(t,x) \;:=\; M_H\left(t, x, \frac{\partial V(t,x)}{\partial x}\right).
$$

In the presence of constraints, the function is not necessarily differentiable, not even continuous, but at least lower semicontinuous under the assumptions of Theorem 3.1.

For the purpose to give a meaning to Hamilton-Jacobi-Bellman partial differential equation (8), we recall one of the definitions of the subdifferential $\partial V(t,x)$ of an extended function $V : \mathbb{R}_+ \times X \mapsto \mathbb{R}_+ \cup \{+\infty\}$

$$(9)\quad \partial V(t,x) := \{(p_t,p_x) \in \mathbb{R} \times X^\star \text{ such that } (p_t,p_x,-1) \in N_{\mathcal{E}p(V)}(t,x,V(t,x))\}.$$

We shall prove that the valuation function is still a solution in the Barron-Jensen/Frankowska sense (discovered independently in [**19, 20**], extending to lower semicontinuous functions the concept of viscosity solutions introduced in [**25**], [**33**] by partial differential equation techniques, and in [**42**], proved with viability tools, following a long series of papers [**37, 38, 39, 40, 41**]).

Theorem 3.2 (Valuation Functions as Solution to Hamilton-Jacobi-Bellman Equations). *If the system is Marchaud, the function $\boldsymbol{b}$ is lower semicontinuous and the function $\boldsymbol{l}$ lower semicontinuous and convex with respect to the control u, then the value function V is the smallest contingent positive solution of the Hamilton-Jacobi-Bellman partial differential equation*

$$\forall\, t > 0,\ \forall\, x \in \overset{\circ}{\boldsymbol{K}}(t) \setminus \boldsymbol{B}(t),\ \forall\, (p_t,p_x) \ \in\ \partial V(t,x),\ \ H(t,x,p_x)\ \leq\ p_t$$

and the condition

$$\forall\, t > 0,\ \forall\, x \in \boldsymbol{B}(t),\ \ V(t,x)\ \leq\ \boldsymbol{b}(t,x).$$

The regulation map governing the evolution of optimal evolutions is then defined by

$$\begin{aligned} R(t,x)\ :=\ &\{u \in U(t,x) \text{ such that } \forall\, (p_t,p_x)\ \in\ \partial V(t,x), \\ &\langle p_x, f(t,x,u)\rangle + \boldsymbol{l}(t,x,u)\ \leq\ p_t\}. \end{aligned}$$

If the system and the transient cost function $\boldsymbol{l}$ are assumed to be Lipschitz, then it is the unique solution in the Barron-Jensen/Frankowska sense

$$\forall\, (p_t,p_x)\ \in\ \partial V(t,x),\ \ H(t,x,p_x)\ =\ p_t$$

and the regulation map is equal to

$$R(t,x)\ :=\ \bigcap_{(p_t,p_x)\in\partial V(t,x)} M_H\,(t,x,p_x)\,.$$

4. Example: The Moskowitz Travel Time Model

We define the *traffic function* $V(t,x) := \int_t^{+\infty} p(x)dx$ as the cumulated number of vehicles from the current position to $+\infty$ (or a finite position) on a one-dimensional road (which is then a decreasing function of the position).

Actually, we shall impose two types of conditions on the traffic function, a *phenomenological law* proposed by Lighthill, Whitham and Richards (see [**43**] and [**52**]) for one-dimensional roads revisited by Moskowitz (see [**47, 48, 49**] for a history of this problem) on the one hand, and, on the other hand, conditions on the traffic functions provided by several types of sensors (Eulerian for fixed sensors, Lagrangian for mobile ones):

(1) **The "Fundamental Diagram"**. Lighthill, Whitham and Richards' theory states that at (T,x), the *traffic density* $p(x) := -\frac{\partial V(t,x)}{\partial x}$ and the *flux* $\frac{\partial V(t,x)}{\partial t}$ of the traffic function are related by the "*fundamental diagram*"

$$H\left(-\frac{\partial V(t,x)}{\partial x}\right) \;=\; \frac{\partial V(t,x)}{\partial t}$$

where the Hamiltonian is independent of (T,x) and is concave and upper semicontinuous with respect to p.

In other words, this means the partial derivatives of the traffic function provide the *traffic states*, i.e., the density-flux pairs, satisfying

$$\left(-\frac{\partial V(t,x)}{\partial x}, \frac{\partial V(t,x)}{\partial t}\right) \;\in\; \mathrm{Graph}(H)$$

which is a Hamilton-Jacobi partial differential equation (see [**36**], for instance).

(2) **Inequality constraints**. Let us consider a cost function $\mathbf{b} : (t,x) \mapsto \mathbf{b}(t,x) \in \mathbb{R} \cup \{+\infty\}$ and its associated set-valued map $t \rightsquigarrow \mathbf{B}(t) := \{x \text{ such that } \mathbf{b}(t,x) \leq +\infty\}$.

We require also that

$$\forall\,(t,x) \in \mathrm{Graph}(\mathbf{B}),\;\; V(t,x) \;\leq\; \mathbf{b}(t,x).$$

This formulation covers many examples which are not detailed in this paper, but in [**8, 9**], [**57**] and the forthcoming book [**6**]. Among them are:

(a) *initial condition (Cauchy)*;
(b) *boundary condition (Dirichlet)*;
(c) *Eulerian conditions* imposing conditions at fixed locations;
(d) *Lagrangian conditions* (see [**56**], for instance) imposing conditions on trajectories.

(3) **The complete model** takes into account the above two requirements:

$$\text{(10)} \qquad \begin{cases} (i) & \forall\, t>0,\; \forall\, x \;\notin\; \mathbf{B}(t),\;\; H\left(-\frac{\partial V(t,x)}{\partial x}\right) \;=\; \frac{\partial V(t,x)}{\partial t}; \\ (ii) & \forall\, t>0,\; \forall\, x \;\in\; \mathbf{B}(t),\;\; V(t,x) \;\leq\; \mathbf{b}(t,x). \end{cases}$$

For solving this problem by viability techniques, we associate with the partial differential equation (10)(i) its *characteristic control system*

$$\text{(11)} \qquad \begin{cases} (i) & x'(t) = u, \\ (ii) & y'(t) \;=\; \mathbf{l}(-u) \end{cases}$$

where $\mathbf{l}$ is the associated *celerity function* defined by the convex Fenchel conjugate

$$\text{(12)} \qquad \mathbf{l}(u) \;:=\; \sup_{p \in \mathrm{Dom}(H)} \;[\langle p,u\rangle + H(p)]$$

where the variable $u \in X$ is regarded as a *celerity*.

Here the density and celerity are regarded as dual variables the duality product of which is the flux, in the same way than in mechanics, position and velocity are dual variable the duality product of which is the power, or, in economics, commodity and price are dual variable the duality product of which is the value of the commodity.

The associated *celerity function* is lower semicontinuous and convex and the fundamental theorem of convex analysis allows the Hamiltonian H to be reconstructed from the convex celerity function (see [**2**], [**10**] and [**53**]) by the formula

$$H(p) \ := \ \inf_u \left(\langle p, u \rangle + \mathbf{l}(-u) \right).$$

For defining the *viability episolution* to problem (10), we introduce the auxiliary control system

$$\left\{ \begin{array}{lrcl} (i) & \tau'(t) & = & -1; \\ (ii) & x'(t) & = & -u(t); \\ (iii) & y'(t) & = & -\mathbf{l}(-u(t)) \text{ where } u(t) \in -\mathrm{Dom}(\mathbf{l}). \end{array} \right. \tag{13}$$

Definition 4.1 (Viability Episolution to the Moskowitz). Let us define the epigraph $\mathcal{E}p(\mathbf{b})$ of the function $\mathbf{b}$. The *viability episolution* V to problem (10), is defined by the following formula

$$V(T,x) \ := \inf_{(T,x,y) \in \mathrm{Capt}_{(13)}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+, \mathcal{E}p(\mathbf{b}))} y. \tag{14}$$

Making explicit the above definition, one can prove that the episolution satisfies the *Daganzo variational principle* for the Moskowitz problem (10), (see [**34, 35**]): we regard $\Delta \in [0,T]$ as a *travel time* from departure time $T-\Delta$ to T. We consider the family of evolutions $x(\cdot) \in \mathcal{O}(T,x)$ starting at departure time $T-\Delta$ at $x(T-\Delta) \in \mathbf{B}(T-\Delta)$. We assign to such an evolution two costs

- the finite cost $\mathbf{b}(T-\Delta, x(T-\Delta))$ on the state
$$x(T-\Delta) \ = \ x - \int_\Delta^T u(\tau) d\tau$$
at time departure time $T-\Delta$;
- the transient cost $\displaystyle\int_{T-\Delta}^T \mathbf{l}(-u(\tau))d\tau$ on the evolution $x(\cdot) \in \mathcal{O}(T,x)$ on the interval $[0,T]$.

We associate with each travel time $\Delta \in [0,T]$ the minimal travel cost on the travel interval $[T-\Delta, T]$ defined by

$$J(T,\Delta;x) \ := \ \mathbf{b}\left(T-\Delta, x - \int_\Delta^T u(\tau)d\tau\right) + \int_{T-\Delta}^T \mathbf{l}(-u(\tau))d\tau.$$

The *Daganzo variational principle* states that the value $V(T,x) = \inf_{\Delta \in [0,T]} J(T,x;\Delta)$ of the viability episolution at (T,x) *minimizes the minimal travel cost with respect to travel time* Δ:

$$V(T,x) =$$
$$\inf_{\Delta \in [0,T]} \ \inf_{(x(\cdot),u(\cdot)) \in \mathcal{O}_{(\mathbf{K},\mathbf{B})}(T,\Delta,x)} \left(\mathbf{b}\left(T-\Delta, x - \int_\Delta^T u(\tau)d\tau\right) + \int_{T-\Delta}^T \mathbf{l}(-u(\tau))d\tau \right).$$

Hence, the traffic function V associated with the function $\mathbf{b}$ is the minimal cost over the controls $u \in -\mathrm{Dom}(\mathbf{l})$ and over the travel time Δ of the sum of the cost $\mathbf{b}(T-\Delta, x-Tu+(T-\Delta)u)$ at initial time $T-\Delta$ of the evolution $t \mapsto x-(T-t)u$ and of the cost $\Delta\mathbf{l}(-u)$ during the travel time Δ at celerity u.

The independence of the Hamiltonian $H(p)$ on (t,x) and the convexity assumptions imply the *Lax-Hopf formula* which states that actually

$$V(T,x) \; = \; \inf_{\Delta\in[0,T]} \; \inf_{u\in-\mathrm{Dom}(\mathbf{l})} \left[\mathbf{b}\left(T-\Delta, x-\Delta u\right) + \Delta\mathbf{l}(-u)\right].$$

This formula implies that if the tube $\mathbf{B}$ defined by $\mathbf{B}(t) := \{x \text{ such that } \mathbf{b}(t,x) < +\infty\}$ has a closed graph and satisfies the Moskowitz property:

$$\forall\, t \geq 0, \;\; \mathbf{B}(t) \; \subset \; \mathbf{B}(0) + t\mathrm{Dom}(\mathbf{l}),$$

then the domain of the traffic functions $V(t,\cdot)$ are equal to

$$\mathrm{Dom}(V(t,\cdot)) \; = \; \mathbf{B}(0) + t\mathrm{Dom}(\mathbf{l}) \tag{15}$$

(see [**5**]).

We can prove that if the Hamiltonian H is concave and upper semicontinuous and if the domain $\mathrm{Dom}(\mathbf{l})$ of the celerity function is compact and if the function $\mathbf{l}$ is bounded above on this domain, then the viability episolution V defined by (14), is the smallest positive satisfying

$$\begin{cases} (i) & \forall\, t>0, \; \forall\, x \; \notin \; \mathbf{B}(t), \;\; \forall\,(p_t,p_x)\in\partial V(t,x), \;\; H(-p_x) \; \leq \; p_t; \\ (ii) & \forall\, t>0, \; \forall\, x \; \in \; \mathbf{B}(t), \;\; V(t,x) \; \leq \; \mathbf{b}(t,x). \end{cases}$$

The regulation map is thus equal to

$$R(t,x) \; := \; \{u \text{ such that } (p_t,p_x)\in\partial V(t,x), \; -p_t - \langle p_x,u\rangle + \mathbf{l}(-u) \; \leq \; 0\}$$

If the function $\mathbf{l}$ is assumed furthermore to be Lipschitz, then it is a solution to the Moskowitz problem (10):

$$\begin{cases} (i) & \forall\, t>0, \; \forall\, x \; \notin \; \mathbf{B}(t), \;\; \forall\,(p_t,p_x)\in\partial V(t,x), \;\; H(-p_x) \; = \; p_t; \\ (ii) & \forall\, t>0, \; \forall\, x \; \in \; \mathbf{B}(t), \;\; V(t,x) \; \leq \; \mathbf{b}(t,x). \end{cases}$$

Setting $\partial_+ H(p) := -\partial(-H)(p)$, the regulation map is given by the following formula

$$R(t,x) \; = \; \bigcap_{(p_t,p_x)\in\partial V(t,x)} \partial_+ H(-p_x) \tag{16}$$

linking the controls governing the optimal travel time evolutions $x(\cdot) \in \mathcal{O}(T,x)$ to the traffic states $(-p_x,p_t) \in \mathrm{Graph}(H)$ provided by subdifferentials $(p_t,p_x) \in \mathrm{Graph}(H)$.

Sketch of the proof. Indeed, for all $(p_t,p_x) \in \partial V(t,x)$, the Hamiltonian associated with this capture basin is defined by

$$-p_t + H(-p_x) \; = \; -p_t + \inf_u \left(\langle p_x, -u\rangle + \mathbf{l}(-u)\right)$$

so that the epigraph of V is a capture basin if and only if for every $(t,x) \notin \mathcal{E}p(\mathbf{b})$, i.e., for every t and x such that $x \notin \mathbf{B}(t)$, and for all $(p_t,p_x) \in \partial V(t,x)$

$$-p_t + \inf_u \left(\langle p_x, -u\rangle + \mathbf{l}(-u)\right) \; \leq \; 0,$$

i.e., if and only if $H(-p_x) \leq p_t$.

When $\mathbf{l}$ is Lipschitz, then the regulation map is defined by the above formula, but knowing furthermore that $H(-p_x) = p_t$. Therefore, the above formula implies that $u \in R(t,x)$ if and only if

$$-H(-p_x) + \mathbf{l}(-u) \; \leq \; \langle -p_x, -u\rangle.$$

Since the functions $-H$ and $\mathbf{l}$ are conjugate, we infer that

$$-u \in \partial(-H)(-p_x) = -\partial_+ H(-p_x),$$

i.e., that

$$M_H(t,x,p_x) := \partial_+ H(-p_x).$$

Therefore, the regulation map can be written

$$R(t,x) = \bigcap_{(p_t,p_x)\in\partial V(t,x)} \partial_+ H(-p_x).$$

Observe that, mathematically, these results hold true not only for 1-dimensional models, but also for n-dimensional ones. More details can be found in [**7, 8, 9**], [**22, 23**] [**30, 31**] and in the forthcoming book [**6**]. □

5. Travel/Arrival Time Set-Valued Maps

In this section, we take into account the dependence of the dynamics and of the environmental constraints not only on arrival time T, but also on travel time $\Delta \in [0,T]$ (and thus, on departure time $T-\Delta$). In this case, $t \in [T-\Delta, T]$ is regarded as the current time and $t-(T-\Delta) \in [0,\Delta]$ as the current travel time:

- the "travel-structured" dynamics $f(t,\delta,x,u)$ and $U(t,\delta,x)$ with which we associate the control system

$$x'(t) = f(t, t-(T-\Delta), x(t), u(t)) \text{ and } u(t) \in U(t, t-(T-\Delta), x(t)); \tag{17}$$

- the "travel-structured" environment map $\mathbf{K}(t,\delta)$ with which we associate the environmental tube $t \rightsquigarrow \mathbf{K}(t, t-(T-\Delta))$.

For any evolution viable in such a tube, we observe that $x(T-\Delta) \in \mathbf{K}(T-\Delta, 0)$ and that $x(T) \in \mathbf{K}(T,\Delta)$

We chose this terminology by analogy with age-structure problems, where T is the time, Δ the age and $T-\Delta$, for this special type of organisms from birth to death.

We also introduce a *departure tube* $\mathbf{B} : \mathbb{R}_+ \rightsquigarrow X$ associating with each departure time d the subset $\mathbf{B}(d) \subset X$ of elements from which start evolutions at departure time $d := T-\Delta$. We assume that

$$\forall\, d := T-\Delta \geq 0, \ \ \mathbf{B}(d) \subset \mathbf{K}(0,d).$$

Definition 5.1 (Travel/Arrival Time Tubes). We define

(1) $\mathcal{O}_{(\mathbf{K},\mathbf{B})} : (T,\Delta,x) \rightsquigarrow \mathcal{O}_{(\mathbf{K},\mathbf{B})}(T,\Delta,c) \in \mathcal{C}(0,+\infty;X)$ of evolutions governed by the controlled travel-structured system (17):

$$x'(t) = f(t, t-(T-\Delta), x(t), u(t)) \text{ and } u(t) \in U(t, t-(T-\Delta), x(t))$$

starting from the departure set $\mathbf{B}(T-\Delta)$ at departure time $T-\Delta$ and viable in the sense that

$$\forall\, t \in [T-\Delta, T], \ \ x(t) \in \mathbf{K}(t, t-(T-\Delta))$$

and arriving at $x = x(T)$ at arrival time T.

(2) The *travel time structure* $\mathbf{A}_{(\mathbf{K},\mathbf{B})} : (T,\Delta) \rightsquigarrow \mathbf{A}_{(\mathbf{K},\mathbf{B})}(T,\Delta)$ as the subset of states $x \in \mathbf{K}(T,\Delta)$ such that there exists at least one evolution $x(\cdot) \in \mathcal{O}_{(\mathbf{K},\mathbf{B})}(T,\Delta,x)$.

As for travel time tubes, we can characterize the graph of a travel/arrival tube as a capture basin:

PROPOSITION 5.2 (Viability Characterization of the Travel/Arrival Time Tubes). *We introduce auxiliary system defined by*

$$(18)\quad \begin{cases} (i) & \tau'(t) \;=\; -1, \\ (ii) & \delta'(t) \;=\; -1, \\ (iii) & x'(t) \;=\; -f(\tau(t),\delta(t),x(t),u(t)) \text{ where } u(t) \in U(\tau(t),\delta(t),x(t)) \end{cases}$$

where $\tau(t) := T-t$ *is the time left to arrival time* T*, the auxiliary environment* $\mathcal{K} :=$ $\mathrm{Graph}(K) \;=\; \{(t,\delta,x)\}$ *and the auxiliary target* $\mathcal{C} := \{(t,0,x)$ such that $x \in \boldsymbol{B}(t)\}$.

Then

$$\mathrm{Graph}(\boldsymbol{A}_{(\boldsymbol{K},\boldsymbol{B})}) \;=\; \mathrm{Capt}_{(18)}(\mathrm{Graph}(\boldsymbol{K}),\mathcal{C}).$$

PROOF. Indeed, to say that (T,Δ,x) belongs to the capture basin $\mathrm{Capt}_{(18)}(\mathcal{K},\mathcal{C})$ of $\mathcal{C}$ viable in $\mathcal{K}$ means that there exists one evolution $t \mapsto (T-t,\Delta-t,\overleftarrow{x}(t))$ to system (18) where $\overleftarrow{x}(\cdot)$ is a solution starting at x at time 0, governed by $\overleftarrow{x}'(t) = -f(,T-t,\Delta-t\overleftarrow{x}(t),\overleftarrow{u}(t))$, and a time $t^\star \geq 0$ such that $(\Delta-t,T-t,\overleftarrow{x}(t))$ is viable in the graph of the tube in the sense that $\overleftarrow{x}(t) \in \mathbf{K}(\Delta-t,T-t)$ on the interval $[0,t^\star]$ until $(T-t^\star,\Delta-t^\star,\overleftarrow{x}(t^\star)) \in \mathcal{C}$. Since $\Delta \leq T$, this condition means that $t^\star = \Delta$ and that $\overleftarrow{x}(\Delta) \in \mathbf{B}(T-\Delta)$. Setting $x(t) := \overleftarrow{x}(T-t)$ and $u(t) := \overleftarrow{u}(T-t)$, we observe that $x(\cdot)$ is a solution governed by differential equation $x'(t) = f(t,t-(T-\Delta),x(t),u(t))$ satisfying $x(T) = \overleftarrow{x}(0) = x$, $x(T-\Delta) = \overleftarrow{x}(\Delta) \in \mathbf{B}(T-\Delta)$ and the viability property

$$\forall\, t \in [T-\Delta,T],\;\; x(t) \;:=\; \overleftarrow{x}(T-\Delta) \;\in\; \mathbf{K}(t,t-(T-\Delta)).$$

This means that $x \in \mathbf{A}_{(\mathbf{K},\mathbf{B})}(T,\Delta)$. □

We now characterize the travel time tube as the unique set-valued solution of a partial differential inclusion.

DEFINITION 5.3 (Travel/Arrival Tube Partial Differential Inclusion). The travel/arrival time tube $\mathbf{A}_{(\mathbf{K},\mathbf{B})}(\cdot,\cdot)$ is the unique tube solution to the set-valued version of the *McKendryk equation* in population dynamics (where travel time Δ plays the role of age and T the role of time) in the following sense:

$$(19)\qquad \begin{cases} (i) & 0 \;\in\; \widehat{D}\mathbf{A}_{(\mathbf{K},\mathbf{B})}(t,\delta,x)(+1,1) - f(t,\delta,x,U(t,\delta,x)), \\ (ii) & 0 \;\in\; D\mathbf{A}_{(\mathbf{K},\mathbf{B})}(t,\delta,x)(+1,+1) \ominus f(t,\delta,x,U(t,\delta,x)) \end{cases}$$

where the *Minkowski difference* $C \ominus A$ is the subset of elements x such that $x + A \subset C$.

The *regulation map* $R(t,\delta,x)$ is defined by

$$R(t,\delta,x) \;:=\; \{u \in U(t,\delta,x) \text{ such that} 0 \;\in\; D\mathbf{A}_{(\mathbf{K},\mathbf{B})}(t,\delta,x)(-1,-1) + f(t,\delta,x,u)\}.$$

This partial differential inclusion (19) can be written in the form

$$\begin{cases} (i) & \exists\; u \;\in\; U(t,\delta,x) \text{ such that } 0 \;\in\; D\mathbf{A}_{(\mathbf{K},\mathbf{B})}(t,\delta,x)(-1,-1) + f(t,\delta,x,u); \\ (ii) & \forall\;\; u \;\in\; U(t,\delta,x) \text{ such that } 0 \;\in\; D\mathbf{A}_{(\mathbf{K},\mathbf{B})}(t,\delta,x)(+1,+1) - f(t,\delta,x,u). \end{cases}$$

Therefore, the controls governing the evolutions starting from $\mathbf{B}(T-\Delta)$ and arriving at x at arrival time T are governed by

$$x'(t) \;=\; f(t,t-(T-\Delta),x(t),u(t)) \;\text{ where }\; u(t) \in R(t,t-(T-\Delta),x(t)).$$

The dual version of this partial differential inclusion translates the dual Frankowska property, involving the do-derivative $D^\star\mathbf{A}_{(\mathbf{K},\mathbf{B})}(t,\delta,x)$ defined as the subset of

$$(p_t,p_\delta) \text{ such that } (p_t,p_\delta,-p_x) \;\in\; N_{\mathrm{Graph}(\mathbf{A}_{(\mathbf{K},\mathbf{B})})}(t,\delta,x).$$

DEFINITION 5.4 (Hamiltonian of the Travel/Arrival PDI). We introduce the Hamiltonian H defined by

$$\begin{array}{c}\forall\, p_x \in X^\star,\ \forall\, (p_t, p_\delta) \in \mathbb{R}^2,\ p_x \in X^\star,\\ H(t,\delta,x,p_t,p_\delta,p_x) \ := \ \inf_{u\in U(t,\delta,x)} \langle p_x, f(t,\delta,x,u)\rangle \,.\end{array}$$

We also introduce the set-valued map

$$U_{\mathbf{K}}(t,\delta,x) \ := \ \{u \in U(t,\delta,x) \text{ such that } f(t,\delta,x,u) \ \in \ D\mathbf{K}(t,\delta,x)(-1,-1)\}\,.$$

The Hamiltonian $H_{\mathbf{K}}$ is defined by

$$\begin{array}{c}\forall\, p_x \in X^\star,\ \forall\, (p_t, p_\delta) \in \mathbb{R}^2,\ p_x \in X^\star,\\ H(t,\delta,x,p_t,p_\delta,p_x) \ := \ \inf_{u\in U_{\mathbf{K}}(t,\delta,x)} \langle p_x, f(t,\delta,x,u)\rangle \,.\end{array}$$

In this case, the regulation map can be written in the form

$$\begin{array}{c}R(t,\delta,x) \ := \ \{u \in U(t,\delta,x) \text{ such that}\\ \forall\, (p_t,p_\delta,p_x) \in \mathrm{Graph}(D^\star \mathbf{A}_{(\mathbf{K},\mathbf{B})}(t,\delta,x)),\ \ \langle p_x, f(t,\delta,x,u)\rangle \ \leq \ p_\delta + p_t\}.\end{array}$$

The *dual Frankowska property* (see (27)) states that

$$\left\{\begin{array}{ll}(i) & \forall\, t>0,\ \forall\, s\geq 0,\ \forall\, x \in (\mathbf{A}_{(\mathbf{K},\mathbf{B})}(t,\delta) \cap (\overset{\circ}{\mathbf{K}}\,(t,\delta)) \setminus \mathbf{B}(t)),\\ & \forall\, (p_t,p_\delta,p_x) \in \mathrm{Graph}(D^\star \mathbf{A}_{(\mathbf{K},\mathbf{B})}(t,\delta,x)),\ p_\delta + p_t \ = \ H(t,\delta,x,p_t,p_\delta,p_x);\\ (ii) & \forall\, t\geq 0,\ \forall\, s\geq 0,\ \forall\, x \in (\mathbf{A}_{(\mathbf{K},\mathbf{B})}(t,\delta) \setminus \overset{\circ}{\mathbf{K}}\,(t,\delta)) \setminus \mathbf{B}(t),\\ & \forall\, (p_t,p_\delta,p_x) \in \mathrm{Graph}(D^\star \mathbf{A}_{(\mathbf{K},\mathbf{B})}(t,\delta,x)),\\ & H(t,\delta,x,p_t,p_\delta,p_x) \ \leq \ p_\delta + p_t \ \leq \ H_{\mathbf{K}}(t,\delta,x,p_t,p_\delta,p_x);\\ (iii) & \forall\, t\geq 0,\ \forall\, s\geq 0,\ \forall\, x \in (\mathbf{A}_{(\mathbf{K},\mathbf{B})}(t,\delta) \setminus \overset{\circ}{\mathbf{K}}\,(t,\delta)) \cap \mathbf{B}(t),\\ & \forall\, (p_t,p_\delta,p_x) \in \mathrm{Graph}(D^\star \mathbf{A}_{(\mathbf{K},\mathbf{B})}(t,\delta,x)),\ p_\delta + p_t \ \leq \ H_{\mathbf{K}}(t,\delta,x,p_t,p_\delta,p_x).\end{array}\right.$$

In this case, the retroaction map R can be written as:

$$\begin{array}{rl} & \forall\, t>0,\ \forall\, s\geq 0, \forall\, x \in (\mathbf{A}_{(\mathbf{K},\mathbf{B})}(t,\delta) \cap \overset{\circ}{\mathbf{K}}\,(t,\delta)) \setminus \mathbf{B}(t),\\ R(t,\delta,x) \ := & \{u \in U(t,\delta,x) \text{ such that } \forall\, (p_t,p_\delta,p_x) \in \mathrm{Graph}(D^\star \mathbf{A}_{(\mathbf{K},\mathbf{B})}(t,\delta,x)),\\ & \langle p_x, f(t,\delta,x,u)\rangle \ = \ p_\delta + p_t\}.\end{array}$$

6. Cournot Set-Valued Maps

The arrival time tube and the arrival/travel map provide, respectively, the sets $\mathbf{P}_{(\mathbf{K},\mathbf{B})}$ and $\mathbf{A}_{(\mathbf{K},\mathbf{B})}(T,\Delta)$ of elements $x \in \mathbf{K}(T,\Delta)$ at which arrive viable evolutions governed by control systems starting from $\mathbf{B}(T-\Delta)$ and viable in the tube $t \rightsquigarrow \mathbf{K}(t, t-(T-\Delta))$ for some travel time $\Delta \in [0,T]$ in the first case and for prescribed time Δ in the second one.

In this section, we ask how to find departure states $\xi \in \mathbf{B}(T-\Delta)$ from which start those viable evolutions arriving at $x = x(T)$ at arrival time T.

We denote this set-valued map under the name of "Cournot maps" because, in 1843, *Augustan Cournot* defined chance as the meeting of several independent causal series: *"A myriad partial series can coexist in time: they can meet, so that a single event, to the production of which several events took part, come from several distinct series of generating causes."*

Definition 6.1 (Cournot Maps). The *Cournot map* associates with any arrival state $x \in \mathbf{A}_{(\mathbf{K},\mathbf{B})}(T,\Delta)$ the subset $\mathrm{Cour}_{(\mathbf{K},\mathbf{B})}(T,\Delta,x)$ of departure states $\xi \in \mathbf{B}(T,\Delta)$ from which starts at least one evolution viable in the tube $t \rightsquigarrow \mathbf{K}(t,t-(T-\Delta))$ on $[0,T]$ until it arrives at $x(T)=x$ at time T.

A pair (T,Δ,x) is an *impact* if the Cournot value $\mathrm{Cour}_{(\mathbf{K},\mathbf{B})}(T,x)$ contains strictly more that one initial state $x_0 \in B$.

As for the other set-valued maps, the graph of the Cournot map is a capture basin and inherits its properties.

Proposition 6.2 (Viability Characterization of Cournot Tubes). *Let us consider the auxiliary system*

$$\text{(20)} \qquad \begin{cases} (i) & \tau'(t) = -1, \\ (ii) & \delta'(t) = -1, \\ (iii) & x'(t) = -f(\tau(t),x(t),u(t)), \\ (iv) & \xi'(t) = 0 \text{ where } u(t) \in U(\tau(t),x(t)), \end{cases}$$

the auxiliary environment $\mathcal{K} := \mathrm{Graph}(K) \times X$ *and the auxiliary target*

$$\mathcal{C} \ := \ \{(t,0,x,x) \text{ where } x \in \boldsymbol{B}(t), \ \ t \geq 0\}\,.$$

The graph of the Cournot map $\mathrm{Cour}_{(\boldsymbol{K},\boldsymbol{B})}$ *is the viable-capture basin of target* $\mathcal{C}$ *viable in* $\mathcal{K}$ *under system (20):*

$$\mathrm{Graph}(\mathrm{Cour}_{(\boldsymbol{K},\boldsymbol{B})}) \ = \ \mathrm{Capt}_{(20)}(\mathcal{K},\mathcal{C}).$$

Proof. To say that (T,Δ,x,ξ) belongs to the capture basin $\mathrm{Capt}_{(20)}(\mathcal{K},\mathcal{C})$ amounts to saying that there exists an evolution $(T-t,\Delta-t,\overleftarrow{x}(t),\xi)$ governed by system (20) starting at $(t,0x,\xi)$ such that $(T-t,\Delta-t,\overleftarrow{x}(t)),\xi$ is viable in $\mathcal{K}$ until it reaches $(T-s,x(s),\xi) \in \mathcal{C}$ at time s. Since $S \leq T$, we deduce that $\Delta - s = 0$. This means that $\overleftarrow{x}(\cdot)$ is an evolution viable in $t \rightsquigarrow \mathbf{K}(T-t,\Delta-t)$ on the interval $[0,\Delta]$, that $\overleftarrow{x}(\Delta) \in \mathbf{B}(\Delta)$ and $x\overleftarrow{x}(\Delta) = \overleftarrow{\xi}(\Delta) = \xi$. Setting $x(t) := \overleftarrow{x}(T-t)$ and $u(t) := \overleftarrow{u}(T-t)$, we infer that $x(T-\Delta) = \overleftarrow{x}(\Delta) \in \mathbf{B}(T-\Delta)$, that $x(T-\Delta)=\xi$, $x(T) = \overleftarrow{x}(0) = x$, that $\xi(T) = \overleftarrow{y}(0) = \xi$, that $x(t) \in \mathbf{K}(t,t-(T-\Delta))$ on the interval $[0,T]$ and that its evolution is governed by

$$x'(t) \ = \ f(t,t-(T-\Delta),x(t),u(t)) \text{ where } u(t) \ \in \ U(t,t-(T-\Delta),x(t)).$$

In other words, ξ belongs to $\mathrm{Cour}_{\mathbf{K}}(T,\Delta,x)$. □

The regulation map associated with the Cournot map is defined by

$$\widehat{R}(t,\delta,x,\xi) \ := \ \{u \in U(t,\delta,x) \mid 0 \in D\mathrm{Cour}_{(\mathbf{K},\mathbf{B})}(t,\delta,x,\xi)(-1,-1,-f(t,x,u))\}.$$

The viable evolutions linking $\xi \in \mathrm{Cour}_{(\boldsymbol{K},\boldsymbol{B})}(T,\Delta,x)$ *to* $x \in \mathrm{Cour}_{(\boldsymbol{K},\boldsymbol{B})}(T)$ *are regulated by*

$$x'(t) \ = \ f(t,t-(T-\Delta),x(t),u(t)) \text{ where } u(t) \ \in \ \widehat{R}(t,t-(T-\Delta),x(t),\xi)$$

starting at $x(T-\Delta) \ = \ \xi$.

7. Appendix: A Viability Survival Kit

We summarize here published ([**1**],[**3, 4**]) and unpublished theorems (to appear in [**6**]) used in this paper. This section presents a few selected statements that are most often used, restricted to capture basins only. Three categories of statements are presented. The first one provides characterizations of capture basins as bilateral fixed points, which are simple, important and are valid without any assumption. The second one provides characterizations in terms of local viability properties and backward invariance, involving topological assumptions on the evolutionary systems. The third one characterizes viability kernels and capture basins under differential inclusions in terms of tangential conditions, which furnishes the regulation map allowing to pilot viable evolutions (and optimal evolutions in the case of optimal control problems) (see [**2**], [**10**] and [**53**]).

7.1. Bilateral fixed point characterization. We consider the maps $(K, C) \mapsto \mathrm{Capt}(K, C)$. The properties of this maps provide fixed point characterizations of viability kernels of the maps $K \mapsto \mathrm{Capt}(K, C)$ and $C \mapsto \mathrm{Capt}(K, C)$ (we refer to [**10**] for more details):

Theorem 7.1 (The Fundamental Characterization of Capture Basins). *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ be an evolutionary system, $K \subset X$ be an environment and $C \subset K$ be a nonempty target.The capture basin $\mathrm{Capt}_{\mathcal{S}}(K, C)$ of C viable in K (see Definition 2.2) is the* unique *subset D between C and K that is both*

1. viable outside C *(and is the largest subset $D \subset K$ viable outside C);*
2. *satisfying* $\mathrm{Capt}_{\mathcal{S}}(K, C) = \mathrm{Capt}_{\mathcal{S}}(K, \mathrm{Capt}_{\mathcal{S}}(K, C))$ *(and is the smallest subset $D \supset C$ to do so):*

i.e., the bilateral fixed point

$$\mathrm{Capt}_{\mathcal{S}}(\mathrm{Capt}_{\mathcal{S}}(K, C), C) = \mathrm{Capt}_{\mathcal{S}}(K, C) = \mathrm{Capt}_{\mathcal{S}}(K, \mathrm{Capt}_{\mathcal{S}}(K, C)). \tag{21}$$

7.2. Viability characterization. It happens that isolated subsets are, under adequate assumptions, backward invariant. Characterizing viability kernels and capture basins in terms of forward viability and backward invariance allows us to use the results on viability and invariance.

Definition 7.2 (Local Viability and Backward Relative Invariance). A subset K is said to be *locally viable* under $\mathcal{S}$ if from any initial state $x \in K$ *there exists at least one evolution* $x(\cdot) \in \mathcal{S}(x)$ and a strictly positive $T_{x(\cdot)} > 0$ such that $x(\cdot)$ is viable in K on the nonempty interval $[0, T_{x(\cdot)}]$. It is a *repeller* under F if all solutions starting from K leave K in finite time.

A subset D is *locally backward invariant relatively to* K if all backward solutions starting from D viable in K are actually viable in K.

If K is itself (backward) invariant, any subset (backward) invariant relatively to K is (backward) invariant. If $C \subset K$ is (backward) invariant relatively to K, then $C \cap \mathrm{Int}(K)$ is (backward) invariant.

Proposition 7.3 (Capture Basins of Relatively Invariant Targets). *Let $C \subset D \subset K$ three subsets of X.*

1. *If D is backward invariant relatively to K, then* $\mathrm{Capt}_{\mathcal{S}}(K, C) = \mathrm{Capt}_{\mathcal{S}}(D, C)$.
2. *If C is backward invariant relatively to K, then* $\mathrm{Capt}_{\mathcal{S}}(K, C) = C$.

Using the concept of backward invariance, we provide a further characterization of capture basins:

THEOREM 7.4 (Characterization of Capture Basins). *Let us assume that $\mathcal{S}$ is upper semicompact, that the environment $K \subset X$ and the target $C \subset K$ are closed subsets satisfying $K \setminus C$ is a repeller ($\mathrm{Viab}_{\mathcal{S}}(K \setminus C) = \emptyset$).*

Then the viable capture basin $\mathrm{Capt}_{\mathcal{S}}(K, C)$ *is the* unique *closed subset D satisfying $C \subset D \subset K$ and*

$$\left\{ \begin{array}{ll} (i) & D \backslash C \text{ is } \textit{locally viable} \text{ under } \mathcal{S}; \\ (ii) & D \textit{ is relatively baclward invariant with respect to } K \text{ under } \mathcal{S}. \end{array} \right. \tag{22}$$

7.3. The regulation map. These theorems, which are valid for any evolutionary systems, paved the way to go one step further when the evolutionary system is a differential inclusion.

We shall use the closed convex hull $T_K^{\star\star}(x)$ of the tangent cone.

Not only does the Viability Theorem provide characterizations of viability kernels and capture basins, but also the *regulation map* $R_D \subset F$ which governs viable evolutions:

DEFINITION 7.5 (Regulation Map). Let us consider three subsets $C \subset D \subset K$ (where the target C may be empty) and a set-valued map $F : X \rightsquigarrow X$.

The set-valued map $R_D : x \in D \rightsquigarrow F(x) \cap T_D^{\star\star}(x) \subset X$ is called the regulation map of F on $D \setminus C$ if

$$\forall\, x \in D \setminus C,\;\; R_D(x) \;:=\; F(x) \cap T_D^{\star\star}(x) \;\neq\; \emptyset. \tag{23}$$

The Viability Theorem implies

THEOREM 7.6 (Tangential Characterization of Capture Basins). *Let us assume that F is Marchaud, that the environment $K \subset X$ and the target $C \subset K$ are closed subsets such that $K \setminus C$ is a repeller ($\mathrm{Viab}_F(K \backslash C) = \emptyset$). Then the viable-capture basin* $\mathrm{Capt}_{\mathcal{S}}(K, C)$ *is the* largest *closed subset D satisfying $C \subset D \subset K$ and*

$$\forall x \in D \backslash C,\;\; F(x) \cap T_D^{\star\star}(x) \;\neq\; \emptyset.$$

Furthermore, for every $x \in D$, there exists at least one evolution $x(\cdot) \in \mathcal{S}(x)$ viable in D until it reaches the target C and all evolutions $x(\cdot) \in \mathcal{S}(x)$ viable in D until they reach the target C are governed by the differential inclusion

$$x'(t) \;\in\; R_D(x(t)).$$

7.4. Frankowska characterizations of the regulation map. These fundamental theorems characterizing viability kernels and capture basins justify a further study of the regulation map and equivalent ways to characterize it. Actually, using the Invariance Theorem, we can go one step further and characterize viability kernels and capture basins in terms of the *Frankowska Property*, stated in two equivalent forms: the *tangential formulation*, expressed in terms of tangent cones, and its *dual version*, expressed in terms of normal cones.

7.4.1. *Tangential Frankowska characterization of the regulation map.* We begin with the case of tangential characterization:

THEOREM 7.7 (Tangential Characterization of Capture Basins). *Let us assume that F is Marchaud and Lipschitz and that the environment $K \subset X$ and the target $C \subset K$ are closed subsets such that $K \setminus C$ is a repeller ($\mathrm{Viab}_F(K \backslash C) = \emptyset$). Then* *the viable-capture basin* $\mathrm{Capt}_{\mathcal{S}}(K,C)$ *is the* unique *closed subset* D *satisfying* $C \subset D \subset K$ *and the* Frankowska property*:*

$$(24) \qquad \begin{cases} (i) & \forall x \in D\backslash C, \;\; F(x) \cap T_D^{\star\star}(x) \;\neq\; \emptyset; \\ (ii) & \forall\, x \in \overset{\circ}{K} \cap D, \;\; -F(x) \;\subset\; T_D^{\star\star}(x); \\ (iii) & \forall x \in \partial K \cap D, \;\; -F(x) \cap T_K(x) \;\subset\; T_D^{\star\star}(x). \end{cases}$$

7.4.2. *Dual Frankowska characterization of the regulation map.* The dual formulation of the Frankowska property involves duality between the finite dimensional vector space X, its dual $X^\star := \mathcal{L}(X,\mathbb{R})$ and its duality pairing $\langle p, x\rangle := p(x)$ on $X^\star \times X$.

Definition 7.8 (Hamiltonian of a Differential Inclusion). We associate with the right-hand side F the *Hamiltonian* $H : X \times X^\star \mapsto \mathbb{R} \cup \{+\infty\}$ defined by

$$(25) \qquad \forall\, x \in X,\; \forall\, p \in X^\star, \;\; H(x,p) \;=\; \inf_{v \in F(x)} \langle p, v\rangle\,.$$

The *constrained Hamiltonian* $H_K : \partial K \times X^\star \mapsto \mathbb{R} \cup \{+\infty\}$ on K is defined by

$$(26) \qquad \forall\, x \in K,\; \forall\, p \in X^\star, \;\; H_K(x,p) \;=\; \inf_{v \in F(x) \cap -T_K(x)} \langle p, v\rangle\,.$$

The function $p \mapsto H(x,p)$ *is concave, positively homogeneous and upper semicontinuous*, as the infimum of continuous affine functions.

The dual version of the tangential conditions characterizing viability kernels and capture basin involve the Hamiltonian of F and "replace" tangent cones by "normal cones": the normal cone

$$N_K(x) \;:=\; T_K(x)^\star \;:=\; \{p \in X^\star \text{ such that } \forall\, v \in T_K(x), \langle p, v\rangle \;\leq\; 0\}$$

to K at x is defined as the polar cone to the tangent cone. Recall that the polar of the normal cone to K at x is equal to the closed convex hull $T_K^{\star\star}(x)$ thanks to the Separation Theorem.

Theorem 7.9 (Dual Characterization of the Regulation Map). *Assume that the images* $F(x)$ *of a set-valued map* F *are compact, convex and not empty on a subset* D

$$\forall\, x \in D, \;\; R_D(x) \;=\; \{v \in F(x) \text{ such that } \forall\, p \in N_D(x), \;\; \langle p, v\rangle \;\leq\; 0\}.$$

If we assume furthermore that for any

$$\forall\, x \in D,\; \forall\, p \in N_D(x,p), \;\; H(x,p) \;\geq\; 0,$$

then

$$\forall\, x \in D, \;\; R_D(x) \;=\; \{v \in F(x) \text{ such that } \forall\, p \in N_D(x), \;\; \langle p, v\rangle \;=\; 0\}.$$

The "dual" version of the tangential characterization of viability kernels is stated in this the following terms :

Theorem 7.10 (Dual Characterization of Capture Basins). *Let us assume that* F *is Marchaud and Lipschitz, that the environment* $K \subset X$ *and the target* $C \subset K$ *are closed subsets such that* $K \setminus C$ *is a repeller (*$\mathrm{Viab}_{\mathcal{S}}(K \setminus C) = \emptyset$*).*

Then the viable-capture basin $\mathrm{Capt}_{\mathcal{S}}(K,C)$ *is the* unique *closed subset satisfying* $C \subset D \subset K$ *and the* dual Frankowska property *(27):*

$$(27) \qquad \begin{cases} (i) & \forall x \in D \cap (\overset{\circ}{K} \backslash C), \forall\, p \in N_D(x), \;\; H(x,p) \;=\; 0; \\ (ii) & \forall x \in D \cap (\partial K \setminus C), \forall\, p \in N_D(x), \quad H(x,p) \;\leq\; 0 \;\leq\; H_K(x,p); \\ (iii) & \forall x \in C \cap \partial K, \forall\, p \in N_D(x), \;\; 0 \;\leq\; H_K(x,p). \end{cases}$$

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Société VIMADES (Viabilité, Marchés, Automatique et Décision), 14, rue Domat, 75005 Paris, France
E-mail address: `aubin.jp@gmail.com`

Cemagref (Laboratoire d'ingénierie des systèmes complexes), 24 avenue des Landais 63172 Aubiere Cedex and CREA, École Polytechnique and CNRS, France
E-mail address: `sophie.martin@cemagref.fr`