SIAM J. CONTROL OPTIM.
Vol. 0, No. 0, pp. 000-000

ANTICIPATING SHOCKS IN THE STATE SPACE: CHARACTERIZING ROBUSTNESS AND BUILDING INCREASINGLY ROBUST EVOLUTIONS*

S. MARTIN† AND I. ALVAREZ‡

Abstract. Given dynamics and constraints, the viability kernel gathers points from which there exists an evolution which remains in the constraint set. In this paper, we aim at providing more information: we introduce and study the robustness map which associates each point of the viability kernel with the maximal size of the unexpected disturbance in the state space the system can support now and in the future while always remaining inside the constraint set. We first characterize the hypograph of the robustness map in terms of a viability kernel of an augmented problem. Then the main mathematical result is that the boundary of this hypograph is a viability domain for a particular augmented problem and that the associated regulation map governs increasingly robust evolutions. Given a time horizon, the problem of finding increasingly robust evolutions which reach a given level of robustness is finally solved by the computation of the reaching time of another augmented problem.

Key words. differential inclusion, control problem, perturbations, robustness, viability kernel

AMS subject classifications. 34A60, 34H05, 93C10, 93C30, 93C73

DOI. 10.1137/16M1061175

1. Introduction.

When dynamical systems are described by ordinary differential equations, a *positively invariant set* has the property that if it contains the system state at some time, then it will contain the system state also in the future [11]. When controlled dynamical systems are concerned, a set is *controlled invariant* if for all initial conditions chosen among its elements, the trajectory remains inside the set by means of a proper control action [11]. In the differential inclusion framework (which encompasses ordinary differential equations and controlled dynamical systems), a viable set gathers states from which at least one solution to the differential inclusion remains inside it. The largest viable set inside a prescribed domain is called the viability kernel [3]. When uncertain systems are considered, the concepts of *robust positively invariant set* [11, 16] or *invariance domain* [3] require that all solutions remain inside the set whatever the perturbation; when a control input is present, the concepts of *robust control invariant set* [11, 16], *discriminating domain* [12], and *guaranteed viability domain* [5] deal with the possibility of finding a control law that governs viable evolutions despite the perturbations.

Regarding uncertainty, viability problems have been studied in the context of stochastic differential inclusions (see, for instance, [9, 20]), which deal with average behaviors, and in the context of tychastic systems defined in [4] when the worst case behavior is considered. In the field of game theory, controller and perturbation can be looked upon as playing a pursuit-evasion game where it is assumed that the perturbation is trying to lead the state outside the constraint set and the controller is trying to prevent it from doing so. Relying on dynamic programming, viscosity solutions of

*Received by the editors February 11, 2016; accepted for publication (in revised form) November 1, 2018; published electronically DATE.
http://www.siam.org/journals/sicon/x-x/M106117.html
†Irstea, UR LISC Laboratoire d'ingénierie des systèmes complexes, Aubière, France (sophie.martin@irstea.fr).
‡Irstea, UR LISC Laboratoire d'ingénierie des systèmes complexes, Aubière, France, and Université Pierre et Marie Curie, LIP6, Paris, France (isabelle.alvarez@irstea.fr).

these games have been proposed (see, for instance, [23]), as well as formulations of the gaming problem in the context of viability theory [12, 13].

Considering the possibility of perturbation occurrence which is not embedded in the dynamics, and which we call *unexpected disturbance*, the achievement of the paper is to provide each point of a viable set with the maximal value of a single disturbance the system can support now and in the future without leaving this viable set. The robustness value of an evolution is linked with the intensity of the disturbance that could happen at any time along the evolution without causing the loss of the system ability to remain in the constraint set. More precisely, in this paper as in [18], we consider unexpected disturbances described by a set-valued map which associates each state of the system with the set that gathers the possible jumps caused by the occurrence of a nominal shock. The robustness question could be addressed by incorporating in the initial dynamical system the possibility of instantaneous jumps and by studying the robust positive invariance or guaranteed viability of these auxiliary dynamical systems (which can be hybrid ones because of the jumps in the state space caused by the unexpected disturbances). But, from a geometric point of view the tolerance to this kind of disturbances is higher when the state of the system is far from the set boundary as underlined in [2] for classification systems, and the map of the distance to the boundary (which can be approximated thanks to an algorithm described in [2]) has been used to propose a family of robustness definitions on evolutions, for instance, the minimum value of the distance along the evolution [1]. In this paper, we pursue this idea and extend the definition of the robustness of an evolution proposed by [1] to consider nominal shocks which depend on the system state. Moreover, since we aim at finding the most robust evolutions, we go beyond the robustness of an evolution to consider the robustness of a state which is the maximal robustness value among all the evolutions starting at this state. Our first result is to characterize the hypograph of this robustness function as the viability kernel of an augmented system. As in [8] for the epigraph of the value function of a discounted infinite horizon optimal control problem, the interest of this result is to allow the computation of the robustness function thanks to algorithms used to compute viable sets (using Euler methods [22], level set approaches [19], or Lagrangian methods [17], for instance). Furthermore, we exhibit sufficient conditions on the set-valued map describing shocks for the boundary of the hypograph of the robustness function to be a viability domain of another augmented system. The main result is then that the regulation map associated with this viability domain allows us to govern increasingly robust evolutions (i.e., evolutions along which the robustness is nondecreasing) which may have the preference of a manager who would take into account the possibility of occurrence of an unexpected disturbance.

The paper is organized as follows. We first introduce the notion of robustness against shocks described by sets of jumps in the state space. We then link the set-valued map describing these shocks with an extended function and exhibit sufficient conditions for this function to have a Lipschitz property. Within the context of evolutions governed by Marchaud differential inclusions, we next use this extended function to define an augmented system and characterize its viability kernel as the hypograph of the robustness function. Then we study properties of the robustness function and of its hypograph. Finally, we define the set of regulations governing increasingly robust evolutions and highlight those which reach a given level of robustness over a given time horizon.

2. Definition of the robustness function. Let $\mathbb{R}^p$ be the system state space. An evolution $x(\cdot) : t \in [0, +\infty[\to x(t) \in \mathbb{R}^p$ is a function of time taking its values in $\mathbb{R}^p$. Let $\mathcal{C}([0, +\infty[; \mathbb{R}^p)$ denote the space of continuous evolutions in the state space $\mathbb{R}^p$. We call an evolutionary system a set-valued map $\mathcal{S} : \mathbb{R}^p \rightsquigarrow \mathcal{C}([0, +\infty[; \mathbb{R}^p)$ which associates any initial state $x \in \mathbb{R}^p$ with a set $\mathcal{S}(x)$ of continuous evolutions $x(\cdot)$ starting from x.

Given $E \subset \mathbb{R}^p$ and $x \in \mathbb{R}^p$, we write $x + E$ for $\{z \in \mathbb{R}^p \mid \exists y \in E$ such that $z = x + y\}$. Let $||\cdot||$ be a norm of $\mathbb{R}^p$, $\mathcal{B}$ is the unit closed ball associated with this norm, and given $x \in \mathbb{R}^p$ and $\delta \geq 0$, $\mathcal{B}(x, \delta) := \{y \in \mathbb{R}^p | \, \|y - x\| \leq \delta\}$. Given $F \subset E \subset \mathbb{R}^p$, we write $E \setminus F$ for $\{x \in E \mid x \notin F\}$ and $\mathrm{Int}_E(F)$ for the interior of F in E:

$$\mathrm{Int}_E(F) := \{x \in F \mid \exists \delta > 0 \text{ such that } (x + \delta\mathcal{B}) \cap E \subset F\} = E \setminus (\overline{E \setminus F}).$$

$\partial_E(F)$ is the boundary of F in E with $\partial_E(F) := \bar{F} \setminus \mathrm{Int}_E(F) = \bar{F} \cap \overline{E \setminus F}$. When E is the whole space, we denote by $\mathrm{Int}(F)$ (respectively, ∂F) the interior (respectively, the boundary) of F.

To deal with uncertainty in the state space, we consider a single shock that can happen anytime and cause a jump in the state space. More precisely, for each state of the system $x \in \mathbb{R}^p$, we consider that the set of reachable states after a shock is defined by $x + mD(x) \subset \mathbb{R}^p$, where $m \geq 0$ is the size of the anticipated disturbances. $D(x)$, which corresponds to possible jumps from x when $m = 1$, is called the set of nominal shocks at x. For example, if $\forall x \in \mathbb{R}^p$, $D(x) = \mathcal{B}$, a nominal shock would provoke a jump of size equal to or smaller than one in the state space.

Definition 2.1. *Let us consider a set-valued map $D : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$. D is a map of nominal shocks if $D(x)$ is a star domain with respect to 0 $\forall x \in \mathbb{R}^p$.*

Actually, when $D(x) \subset \mathbb{R}^p$ is a star domain with respect to 0, the set of reachable states after a shock $x + mD(x) \subset \mathbb{R}^p$ increases with the size $m \geq 0$ of the anticipated disturbances.

Let us consider now a constraint set $K \subset \mathbb{R}^p$, and for any $x \in \mathbb{R}^p$ the set $\mathcal{Q}_K(x)$ of evolutions starting at x that remain in K:

$$\mathcal{Q}_K(x) = \{x(\cdot) : [0, +\infty[\to \mathbb{R}^p \mid x(0) = x \text{ and } \forall t \geq 0,\ x(t) \in K\}. \tag{2.1}$$

Definition 2.2 (viable evolutions). *We shall say that an evolution $x(\cdot) : t \in [0, +\infty[\to x(t) \in \mathbb{R}^p$ is viable in K if $x(\cdot) \in \mathcal{Q}_K(x(0))$.*

Then given $x_0 \in \mathbb{R}^p$, the set of evolutions starting at x_0 governed by $\mathcal{S}$ and viable in K, is $\mathcal{S}(x_0) \cap \mathcal{Q}_K(x_0)$.

Definition 2.3. *Given an evolutionary system $\mathcal{S} : \mathbb{R}^p \rightsquigarrow \mathcal{C}([0, +\infty[; \mathbb{R}^p)$, a map of nominal shocks $D : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$, and a constraint set $K \subset \mathbb{R}^p$,*

- *the robustness in K of an evolution $x(\cdot) \in \mathcal{S}(x(0))$ against a single shock described by D, denoted by $\rho^{\#}_{\mathcal{S},D,K}(x(\cdot)) \in [0, +\infty[\cup\{\pm\infty\}$, is the supremum of all $m \geq 0$ such that, for every $T \geq 0$, every $x_T \in x(T) + mD(x(T))$, there exists an evolution $y(\cdot) \in \mathcal{S}(x_T)$ that is viable in K (with the convention $\sup(\emptyset) := -\infty$);*
- *the robustness (against a single shock) of a state $x \in \mathbb{R}^p$ is the supremum of the robustness values of all the evolutions starting at x. The robustness function $\rho : \mathbb{R}^p \to [0, +\infty[\cup\{\pm\infty\}$ is defined as follows:*

$$\rho(x_0) := \sup_{x(\cdot) \in \mathcal{S}(x_0)} \rho^{\#}_{\mathcal{S},D,K}(x(\cdot)). \tag{2.2}$$

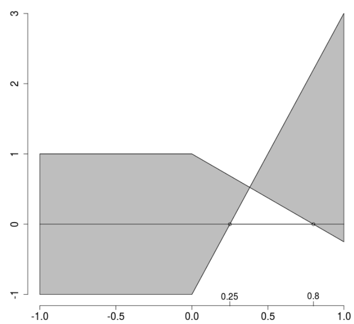

FIG. 2.1. *The gray area is the graph of U for $x \in [-1;1]$.*

To illustrate the above definitions, let us consider the one-dimensional differential inclusion:

$$x'(t) \quad \in \quad U(x(t)) \tag{2.3}$$

with

$$\begin{cases} U(x) & = & [\min(1-5x/4, 4x-1); \max(1-5x/4, 4x-1)] \text{ if } x \in [0; +\infty[\\ & = & [-1;1] \text{ otherwise.} \end{cases} \tag{2.4}$$

Figure 2.1 displays the sets $U(x)$ for $x \in [-1;1]$.

Let the constraint set K be the line segment $[-1;1]$ and let the map of nominal shocks D associate with all $x \in \mathbb{R}$ the closed one-dimensional unit ball $[-1;1]$.

When $x = -1$ or $x = 1$, $0 \in U(x)$, so K is viable and coincides with its viability kernel.

From points x_0 within $[-1;0.25] \cup [0.8;1]$, the constant evolutions are viable and the robustness in K of these evolutions against a single shock described by D equals $1 - |x_0| = d(x_0)$. Consequently, for all points $x_0 \in [-1;0.25] \cup [0.8;1]$ the value of the robustness function equals $1 - |x_0|$.

Hence, inside $([-1;0.25] \cup [0.8;1]) \cap [0, +\infty[$, along a viable evolution, functions associating with time the x-coordinate and the robustness display opposite variations. In particular, choosing a positive value for the variation of x on a given time interval causes a decrease of robustness during this period.

When $x_0 \in]0.25;0.8[$, $\forall x(.) \in \mathcal{S}(x_0)$ and $\forall 0 < \epsilon < 0.8 - x_0$, one has $x'(t) \geq \min(4x_0 - 1, 1 - 5(0.8-\epsilon)/4) > 0$ from $t = 0$ until $x(t) = 0.8 - \epsilon$. So, there exists $T > 0$ such that $x(t) \geq 0.8 - \epsilon$ when $t \geq T$. Hence $\rho(x_0) = \rho(0.8) = 1 - 0.8$ $\forall x_0 \in]0.25;0.8[$ and ρ is discontinuous at 0.25. Figure 2.2 displays the graphs of d and ρ for $x \in [-1;1]$.

3. Associating a set-valued map describing shocks with an extended function. Given a map of nominal shocks $D : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$ and a closed subset $K \subset \mathbb{R}^p$, we consider the extended function which associates with $x \in \mathbb{R}^p$ the value $\max\{m \geq 0 \mid x + mD(x) \subset K\}$. This section describes sufficient conditions on D for this function to be Lipschitz on K. In the particular case when $\forall x \in \mathbb{R}^p$, $D(x) = \mathcal{B}$, this function is the distance to the boundary of K on which emphasis is put in [2, 1] to consider robustness issues.

In the following, we consider particular maps of nominal shocks.

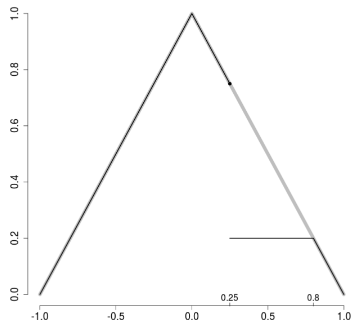

FIG. 2.2. *The graph of the distance function to the boundary of $[-1;1]$ for $x \in [-1;1]$ is drawn with a bold gray line. The graph of the robustness function for $x \in [-1;1]$ is drawn with a black line.*

DEFINITION 3.1. *Let us denote by $\mathcal{M}$ the set gathering the maps of nominal shocks $D : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$ satisfying*

- *$\forall x \in \mathbb{R}^p$, $D(x)$ is compact and convex,*
- *$\exists b > 0$ such that $\forall x \in \mathbb{R}^p$, $b\mathcal{B} \subset D(x)$,*
- *D is Lipschitz.*[1]

Given $D \in \mathcal{M}$ and $x \in \mathbb{R}^p$, since $D(x)$ contains 0 and is compact and convex with nonempty interior in $\mathbb{R}^p$, we can associate with $D(x)$ the Minkowski functional $\mathcal{G}_{D(x)} : \mathbb{R}^p \to [0, +\infty[$ defined by

$$\mathcal{G}_{D(x)}(z) := \inf\{\lambda > 0 \mid z \in \lambda D(x)\}, \tag{3.1}$$

which verifies

- it is subadditive since $D(x)$ is convex,
- $\mathcal{G}_{D(x)}(z) \leq \frac{\|z\|}{b}$ since $b\mathcal{B} \subset D(x)$,
- when $\mu \geq 0$, $\mathcal{G}_{D(x)}(\mu z) = \mu \mathcal{G}_{D(x)}(z)$,
- if $\mathcal{G}_{D(x)}(z) = 0$, then $z = 0$, since $D(x)$ is bounded.

We associate with this Minkowski functional the function $d_{D(x)}$ defined by

$$d_{D(x)}(z_1, z_2) := \mathcal{G}_{D(x)}(z_1 - z_2). \tag{3.2}$$

LEMMA 3.2. *Let us consider $C \subset \mathbb{R}^p$ a compact and convex subset of $\mathbb{R}^p$. Assume that there exists $b > 0$ such that $b\mathcal{B} \subset C$. Let $\mathcal{G}_C$ be the Minkowski functional associated with C. Then $\forall \epsilon > 0$ and $\forall x \in C + \epsilon\mathcal{B}$, $\mathcal{G}_C(x) \leq 1 + \frac{\epsilon}{b}$.*

Proof. If $x \in C + \epsilon\mathcal{B}$, there exist $y \in C$ and $z \in \mathbb{R}^p$ such that $x = y + z$ with $\|z\| \leq \epsilon$. From the triangle inequality , $\mathcal{G}_C(x) \leq \mathcal{G}_C(y) + \mathcal{G}_C(z)$. $\mathcal{G}_C(y) \leq 1$ since $y \in C$ and $\mathcal{G}_C(z) \leq \frac{\|z\|}{b}$ since $b\mathcal{B} \subset C$. ∎

PROPOSITION 3.3. *Let us consider $D \in \mathcal{M}$ with Lipschitz constant k; then $\forall x, y, z \in \mathbb{R}^p$,*

$$|\mathcal{G}_{D(x)}(z) - \mathcal{G}_{D(y)}(z)| \leq \frac{k}{b^2} ||x - y|| ||z||.$$

[1]A set-valued map F is said to be Lipschitz if there exists a constant $\lambda > 0$ such that $\forall x, y \in \mathbb{R}^p$, $F(x) \subset F(y) + \mathcal{B}(0, \lambda||x - y||)$.

Proof. If $z = 0$, $\mathcal{G}_{D(x)}(z) = \mathcal{G}_{D(y)}(z) = 0$. Otherwise, if $z \neq 0$ and $\mathcal{G}_{D(x)}(z) \geq \mathcal{G}_{D(y)}(z)$, then

$$\begin{aligned}\mathcal{G}_{D(x)}(z) - \mathcal{G}_{D(y)}(z) &= \left(\mathcal{G}_{D(x)}\left(\frac{z}{\mathcal{G}_{D(y)}(z)}\right) - 1\right)\mathcal{G}_{D(y)}(z)\\ &\leq \left(1 + \frac{k\|x-y\|}{b} - 1\right)\mathcal{G}_{D(y)}(z)\\ &\leq \frac{k\|x-y\|}{b}\frac{\|z\|}{b},\end{aligned}$$

where the first inequality comes from Lemma 3.2 and the second from the b-ball being in $D(y)$. ∎

Proposition 3.4. *Let us consider $D \in \mathcal{M}$ and $K \subset \mathbb{R}^p$ a nonempty compact subset of $\mathbb{R}^p$. Let ∂K be the boundary of K in $\mathbb{R}^p$. Let $d_{D,K} : \mathbb{R}^p \to [0, +\infty[\cup\{-\infty\}$ be the extended function defined by*

$$d_{D,K}(x) := \begin{cases} \min_{z\in\partial K}(d_{D(x)}(x,z)) & \textit{if } x \in K,\\ -\infty & \textit{otherwise.}\end{cases} \tag{3.3}$$

Then there exists $\tilde{k} > 0$ such that $\forall x, y \in K$, $|d_{D,K}(x) - d_{D,K}(y)| \leq \tilde{k}||x-y||$. Moreover,

$$d_{D,K}(x) = \max\{m \geq 0 \mid x + mD(x) \subset K\} \tag{3.4}$$

with the convention $\max(\emptyset) = -\infty$.

Proof. Since K is compact, let us introduce $R := \max_{x,y\in K}||x-y|| < +\infty$. Let us consider $x, y \in K$ and $z \in \partial K$,

$$\begin{array}{rcll} d_{D(x)}(x,z) & \leq & d_{D(x)}(x,y) + d_{D(x)}(y,z) & \text{triangle inequality}\\ & \leq & \frac{||x-y||}{b} + \mathcal{G}_{D(x)}(y-z) & \\ & \leq & \frac{||x-y||}{b} + \frac{k}{b^2}||x-y||||y-z|| + \mathcal{G}_{D(y)}(y-z) & \text{from Proposition 3.3}\\ & \leq & (\frac{1}{b} + \frac{kR}{b^2})||x-y|| + d_{D(y)}(y,z)\ . & \end{array} \tag{3.5}$$

Then $\min_{z\in\partial K} d_{D(x)}(x,z) \leq \tilde{k}||x-y|| + d_{D(y)}(y,z)$, where $\tilde{k} := \frac{1}{b} + \frac{kR}{b^2}$, and $d_{D,K}(x) \leq \tilde{k}||x-y|| + \min_{z\in\partial K} d_{D(y)}(y,z)$, so $d_{D,K}(x) \leq \tilde{k}||x-y|| + d_{D,K}(y)$. With the same reasoning, we get $d_{D,K}(y) \leq \tilde{k}||x-y|| + d_{D,K}(x)$ and $|d_{D,K}(x) - d_{D,K}(y)| \leq \tilde{k}||x-y||$.

Let us consider $x \in K$ and $m \geq 0$ such that $x + mD(x) \subset K$. Since $x + mD(x) = \{y \in \mathbb{R}^p \mid \mathcal{G}_{D(x)}(y-x) \leq m\} = \{y \in \mathbb{R}^p \mid d_{D(x)}(x,y) \leq m\}$, so

$$\{y \in \mathbb{R}^p \mid d_{D(x)}(x,y) \leq m\} \subset K\ .$$

Consequently, $\forall z \in \partial K$, $d_{D(x)}(x,z) \geq m$, $d_{D,K}(x) = \min_{z\in\partial K} d_{D(x)}(x,z) \geq m$, and $d_{D,K}(x) \geq \max\{m \geq 0 \mid x + mD(x) \subset K\}$.

Conversely, if $\tilde{m} > \max\{m \geq 0 \mid x + mD(x) \subset K\}$, then $\exists z_x \in D(x)$ such that $x + \tilde{m}z_x \notin K$. Since $x \in K$, there exists $0 \leq \lambda < \tilde{m}$ such that $z := x + \lambda z_x \in \partial K$ and $d_{D(x)}(x,z) \leq \lambda < \tilde{m}$. So, $d_{D,K}(x) < \tilde{m}$. ∎

Remark. Given a nonempty compact subset $K \subset \mathbb{R}^p$, if $\forall x \in \mathbb{R}^p$, $D(x) = \mathcal{B}$, then $d_{D,K}(\cdot)$ is the usual distance to the boundary of K for points inside K:

$$d_{D,K}(x) = \begin{cases} \min_{y\in\partial K}||x-y|| & \text{if } x \in K,\\ -\infty & \text{otherwise.}\end{cases} \tag{3.6}$$

4. Viability characterization of the robustness function against a single shock. From now on, we consider evolutionary systems which are associated with differential inclusions. Let $F : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$ be a set-valued map associating with any state $x \in \mathbb{R}^p$ the set $F(x) \subset \mathbb{R}^p$ of velocities available at state x. It defines the differential inclusion (see [15, 6, 7, 3, 21])

$$x'(t) \in F(x(t)). \tag{4.1}$$

Let us denote by $W^{1,1}([0,+\infty[,\mathbb{R}^p)$ the set of absolutely continuous functions on $\mathbb{R}^p$. We recall that a continuous function $x(\cdot) : [0,+\infty[\to \mathbb{R}^p$ is said to be absolutely continuous if there exists a locally integrable function v such that

$$\forall t, s \in [0,+\infty[,\ \int_t^s v(\tau)d\tau = x(s) - x(t).$$

In this case, for almost all $t \in [0,+\infty[,\ x'(t) := v(t)$ and we shall say that $x'(\cdot)$ is the weak derivative of the function $x(\cdot)$.

Given a set-valued map $F : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$, we denote by $\mathrm{Dom}(F)$ the domain of F that is the subset of $\mathbb{R}^p$ whose elements x are such that $F(x) \neq \emptyset$.

Given a set-valued map F, we can define the evolutionary system $\mathcal{S} : \mathbb{R}^p \rightsquigarrow W^{1,1}([0,+\infty[,\mathbb{R}^p)$ which associates with $x_0 \in \mathbb{R}^p$ the set of evolutions starting at x_0 and governed by the differential inclusion (4.1):

$$\mathcal{S}(x_0) = \left\{ \begin{array}{c} x(\cdot) : [0,+\infty[\to \mathrm{Dom}(F) \in W^{1,1}([0,+\infty[,\mathbb{R}^p) \text{ such that} \\ x(0) = x_0 \text{ and for almost all } t \geq 0,\ x'(t) \in F(x(t)) \end{array} \right\}. \tag{4.2}$$

We shall say that $\mathcal{S}$ is the evolutionary system associated with the set-valued map F. Let us recall the definitions of viable set and viability kernel from [3].

Definition 4.1 (viable set and viability kernel). *Given a set-valued map F and a subset $K \subset Dom(F)$, let $\mathcal{S}$ be the evolutionary system associated with F and let $\mathcal{Q}_K$ be the map associating with $x \in K$ the set of evolutions starting at x and viable in K* (2.1).

We shall say that K is viable under $\mathcal{S}$ if for any initial state x_0 in K, there exists $x(\cdot) \in \mathcal{S}(x_0)$ which is viable in K, that is, $x(\cdot) \in \mathcal{S}(x_0) \cap \mathcal{Q}_K(x_0)$.

We shall say that the largest closed subset of K viable under $\mathcal{S}$ denoted by $Viab_{\mathcal{S}}(K)$ (which may be empty) is the viability kernel of K for $\mathcal{S}$.

Given a set-valued map $F : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$ and an upper semicontinuous function $\gamma : \mathrm{Dom}(F) \to [0,+\infty[$ with linear growth, we consider the auxiliary differential inclusion defined by

$$\left\{ \begin{array}{lcl} x'(t) & \in & F(x(t)), \\ y'(t) & \in & [0; \gamma(x(t))]. \end{array} \right. \tag{4.3}$$

We denote by $\mathcal{S}_1$ the evolutionary system associated with the differential inclusion (4.3). We recall that a set-valued map $F : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$ is a Marchaud map if it is nontrivial, is upper semicontinuous, and has compact convex images and linear growth, that is, there exists $c > 0$ such that $\forall x \in \mathrm{Dom}(F)$, $\|F(x)\| \leq c(\|x\|+1)$, with $\|F(x)\| = \sup_{y\in F(x)} \|y\|$. A set-valued map $F : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$ is upper semicontinuous if it satisfies, for any $x_0 \in \mathbb{R}^p$,

$$\forall \epsilon > 0,\ \exists \eta > 0 \text{ such that } \forall x \in \mathbb{R}^p,\ \|x - x_0\| \leq \eta \Rightarrow F(x) \subset F(x_0) + \epsilon\mathcal{B}.$$

We recall that the hypograph of an extended function $v : \mathbb{R}^p \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is the set of pairs $(x, y) \in \mathbb{R}^p \times \mathbb{R}$ satisfying $v(x) \geq y$. It is denoted by $\mathcal{H}yp(v)$. The domain of the extended function v defined by $\mathrm{Dom}(v) := \{x \in X \mid -\infty < v(x) < +\infty\}$ is the set of elements on which the function is finite. The graph of v denoted by $\mathrm{Graph}(v)$ is the set of pairs $(x, y) \in \mathrm{Dom}(v) \times \mathbb{R}$ satisfying $v(x) = y$.

Theorem 4.2 (viability characterization of the robustness function against a single shock). *Given a Marchaud set-valued map F, let $\mathcal{S}$ be the evolutionary system associated with F and let $\mathcal{S}_1$ be the auxiliary evolutionary system associated with the differential inclusion* (4.3). *Given K a compact subset of $Dom(F)$ such that $Viab_{\mathcal{S}}(K) \neq \emptyset$ and a map of nominal shocks $D \in \mathcal{M}$, let us omit the subscripts in $d_{D, Viab_{\mathcal{S}}(K)}$, $d(x) := \max\{m \geq 0 \mid x + mD(x) \subset Viab_{\mathcal{S}}(K)\}$. Let $\mathcal{H}yp(d)$ be the hypograph of the extended function d. Then*

$$\forall x \in Dom(F),\ \rho(x) = \sup_{(x,y)\in Viab_{\mathcal{S}_1}(\mathcal{H}yp(d))} y. \tag{4.4}$$

Moreover,

$$Dom(\rho) = Dom(d) = Viab_{\mathcal{S}}(K), \tag{4.5}$$

$$\mathcal{H}yp(\rho) = Viab_{\mathcal{S}_1}(\mathcal{H}yp(d)) \tag{4.6}$$

and ρ is upper semicontinuous.

Proof. Since K is closed and F is Marchaud, from the viability theorem [3, p. 121], $\mathrm{Viab}_{\mathcal{S}}(K)$ exists, is closed, and is equal to the subset of states $x \in \mathrm{Dom}(F)$ such that at least one element of $\mathcal{S}(x)$ is viable in K.

From Proposition 3.4, it follows that d is Lipschitz on $\mathrm{Viab}_{\mathcal{S}}(K)$, which is compact, and by definition, d is $-\infty$ outside of that set. Consequently, d is upper semicontinuous and its hypograph is closed.

Since F is Marchaud and γ is upper semicontinuous with linear growth on $\mathrm{Dom}(F)$, the set-valued map of the right side of (4.3) is also Marchaud. Moreover, since $\mathcal{H}yp(d)$ is closed, from the viability theorem [3, p. 121], $\mathrm{Viab}_{\mathcal{S}_1}(\mathcal{H}yp(d))$ exists, is closed, and is equal to the subset of initial states such that at least one evolution starting from them and governed by (4.3) is viable in $\mathcal{H}yp(d)$.

If $(x, y) \in \mathrm{Viab}_{\mathcal{S}_1}(\mathcal{H}yp(d))$, then there exist an evolution $x(\cdot) \in \mathcal{S}(x)$ and an evolution $y(\cdot) : [0, +\infty[\to \mathbb{R}$ such that $(x(t), y(t)) \in \mathcal{H}yp(d)$ $\forall t \geq 0$, i.e., such that, $\forall t \geq 0$, $y(t) \leq d(x(t))$. Since from (4.3), for almost all $t \geq 0$, $y'(t) \geq 0$, $\forall t \geq 0$, $y(t) \geq y$ and then $\forall t \geq 0$, $y \leq d(x(t))$ and $y \leq \inf_{t\geq 0} d(x(t))$. Moreover, let us take $m := \inf_{t\geq 0} d(x(t))$, $\forall t \geq 0$, $x(t) + mD(x(t)) \subset \mathrm{Viab}_{\mathcal{S}}(K)$ and $x(\cdot) \in \mathcal{S}(x)$ is robustly viable against a single shock of maximal size m and $m \leq \rho(x)$. Consequently, $y \leq \rho(x)$ and then

$$\tilde{y}(x) := \sup_{(x,y)\in \mathrm{Viab}_{\mathcal{S}_1}(\mathcal{H}yp(d))} y \leq \rho(x)\ . \tag{4.7}$$

For proving the opposite inequality, let us take $\mu < \rho(x)$. By definition of the supremum, there exists an evolution $x(\cdot) \in \mathcal{S}(x)$ such that $\mu \leq \rho^{\#}_{\mathcal{S},D,K}(x(\cdot))$. Hence $\forall T \geq 0$, $x(\cdot)$ is robustly viable against a single shock of maximal size μ at time T. This implies that $\forall p \in \mu D(x(T))$, there exists a viable evolution $\hat{x}(\cdot) \in \mathcal{S}(x(T) + p) \cap \mathcal{Q}_K(x(T)+p)$. Consequently, $x(T)+p \in \mathrm{Viab}_{\mathcal{S}}(K)$ and $x(T)+\mu D(x(T)) \subset \mathrm{Viab}_{\mathcal{S}}(K)$, then $\mu \leq d(x(T))$ $\forall T \geq 0$. This implies that $(x(\cdot), \mu(\cdot) = \mu) \in \mathcal{S}_1(x, \mu)$ is viable in $\mathcal{H}yp(d)$, $(x, \mu) \in \mathrm{Viab}_{\mathcal{S}_1}(\mathcal{H}yp(d))$ and therefore $\mu \leq \tilde{y}(x)$. So, $\tilde{y}(x) \geq \rho(x)$.

Moreover, let us show that $H := \mathrm{Viab}_{\mathcal{S}_1}(\mathcal{H}yp(d))$ is a hypograph. Given $(x,y) \in H$, there exists an evolution $(x(\cdot), y(\cdot))$ governed by (4.3) starting at (x,y) and remaining in H. Let us take $y' \leq y$, and consider the evolution $(x(\cdot), y(\cdot) - y + y')$. It is also governed by (4.3) since the dynamics (4.3) do not depend on $y(t)$. Moreover, $\forall t \geq 0$, $y(t) - y + y' \leq y(t) \leq d(x(t))$. Consequently, the evolution $(x(\cdot), y(\cdot) - y + y')$ remains in H. So $(x, y') \in H$ and H is a hypograph. Finally, $\mathcal{H}yp(\rho)$ is closed and then ρ is upper semicontinuous. □

Let $\mathcal{S}$ be an evolutionary system on $\mathbb{R}^p$, and a closed subset $F \subset \mathbb{R}^p$ is called a viability domain for $\mathcal{S}$ if there exists a closed subset $E \subset \mathbb{R}^p$ such that $F \subset E$ and $F = \mathrm{Viab}_{\mathcal{S}}(E)$.

Corollary 4.3.

$$\forall \bar{y} \geq 0,\ \mathcal{H}yp(\rho) \cap \{(x,y) \in \mathbb{R}^p \times \mathbb{R} | y = \bar{y}\} \tag{4.8}$$

is a viability domain for (4.3). *Moreover,*

$$\{x \in K \mid \rho(x) \geq \bar{y}\} = Viab_{\mathcal{S}}(K_{\bar{y}}), \tag{4.9}$$

where $K_{\bar{y}} := \{x \in K \mid x + \bar{y}D(x) \subset K\} = \{x \in K \mid d(x) \geq \bar{y}\}$.

Proof. Let us consider the set-valued map $F_0 : \mathbb{R}^p \times \mathbb{R} \rightsquigarrow \mathbb{R}^p \times \mathbb{R}$ defined by $F_0(x,y) = F(x) \times \{0\}$ and the associated differential inclusion:

$$\left\{ \begin{array}{lcl} x'(t) & \in & F(x(t)), \\ y'(t) & = & 0. \end{array} \right. \tag{4.10}$$

Let $\mathcal{S}_{1,F_0}$ be the evolutionary system associated with F_0. Equation (4.10) is a particular case of differential inclusions described by (4.3) with $\forall x \in \mathbb{R}^p$, $\gamma(x) = 0$, and then $\forall x \in \mathbb{R}^p$, $\mathcal{S}_{1,F_0}(x) \subset \mathcal{S}_1(x)$. Moreover, from Theorem 4.2, $\mathcal{H}yp(\rho) = \mathrm{Viab}_{\mathcal{S}_{1,F_0}}(\mathcal{H}yp(d))$, and y remains constant along the evolutions governed by F_0. So, $\forall \bar{y} \geq 0$,

$$\mathcal{H}yp(\rho) \cap \{(x,y) \in \mathbb{R}^p \times \mathbb{R} | y = \bar{y}\} = \mathrm{Viab}_{\mathcal{S}_{1,F_0}} \left(\mathcal{H}yp(d) \cap \{(x,y) \in \mathbb{R}^p \times \mathbb{R} | y = \bar{y}\}\right) \tag{4.11}$$

and

$$\{x \in K \mid \rho(x) \geq \bar{y}\} = \mathrm{Viab}_{\mathcal{S}}(K_{\bar{y}}). \tag{4.12}$$

Then $\forall \bar{y} \geq 0$, $\mathcal{H}yp(\rho) \cap \{(x,y) \in \mathbb{R}^p \times \mathbb{R} | y = \bar{y}\}$ is a viability domain for (4.10).

But, since $\forall x \in \mathbb{R}^p$, $\mathcal{S}_{1,F_0}(x) \subset \mathcal{S}_1(x)$, $\forall \bar{y} \geq 0$, $\mathcal{H}yp(\rho) \cap \{(x,y) \in \mathbb{R}^p \times \mathbb{R} | y = \bar{y}\}$ is also a viability domain for (4.3). □

5. Properties of the robustness function and of its hypograph. In this section, we assume that the conditions of Theorem 4.2 are satisfied: we consider an evolutionary system $\mathcal{S}$ associated with a Marchaud map $F : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$, a compact subset $K \subset \mathrm{Dom}(F)$ such that $\mathrm{Viab}_{\mathcal{S}}(K) \neq \emptyset$, a map of nominal shocks $D : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p \in \mathcal{M}$, the associated extended function $d : \mathbb{R}^p \to [0, +\infty[\cup \{-\infty\}$, $d := d_{D,\mathrm{Viab}_{\mathcal{S}}(K)}$ (3.4) which is Lipschitz continuous on $\mathrm{Viab}_{\mathcal{S}}(K)$ from Proposition 3.4, and the robustness function $\rho : \mathbb{R}^p \to [0, +\infty[\cup \{-\infty\}$ (2.2). The hypograph of the robustness function ρ is a subset of $\mathbb{R}^{p+1}$ and it is closed from Theorem 4.2. Let us consider the closed subset

$$E := \partial \mathcal{H}yp(\rho) \cap (\mathbb{R}^p \times [0, +\infty[). \tag{5.1}$$

Let us set

$$E_1 := \{(x, y) \in E | \rho(x) = d(x) \text{ and } y = \rho(x)\},$$
$$E_2 := \{(x, y) \in E | \rho(x) = d(x) \text{ and } y < \rho(x)\},$$
$$E_3 := \{(x, y) \in E | \rho(x) < d(x) \text{ and } y = \rho(x)\},$$
$$E_4 := \{(x, y) \in E | \rho(x) < d(x) \text{ and } y < \rho(x)\},$$
$$E_{24} := E_2 \cup E_4.$$

Obviously, $E = E_1 \cup E_2 \cup E_3 \cup E_4$, $E_1 \cup E_3 = \text{Graph}(\rho)$ and $E_i \cap E_j = \emptyset$ when $i \neq j$. $E_1 = E \cap \partial \mathcal{H}yp(d)$ is closed and $E_2 \cup E_3 \cup E_4 = E \cap \text{Int}(\mathcal{H}yp(d))$.

Moreover, $\partial_{\mathcal{H}yp(\rho)}(E) = \partial \text{Viab}_{\mathcal{S}}(K) \times \{0\} \subset E_1$. Consequently, since E_1 is closed, then

$$E_{24} \setminus \text{Int}_E(E_{24}) = E_{24} \cap \overline{E_3}. \tag{5.2}$$

Figure 5.1 displays sets E, E_1, E_2, and E_3 ($E_4 = \emptyset$) for the illustrative example of section 2.

Since F is a Marchaud map and K is compact, we define $M := \sup_{x \in K} \sup_{x' \in F(x)} ||x'|| < +\infty$. Let k be the Lipschitz constant of d on $\text{Viab}_{\mathcal{S}}(K)$. Let us consider the auxiliary differential inclusion on $E \subset \mathbb{R}^p \times [0, +\infty[$:

$$\left\{ \begin{array}{lcll} x'(t) & \in & F(x(t)), & \\ y'(t) & = & v(t) \text{ where } v(t) \in [0; 2kM] & \text{if } (x, y) \in E \setminus \text{Int}_E(E_{24}) \\ & = & 2kM & \text{if } (x, y) \in \text{Int}_E(E_{24}). \end{array} \right. \tag{5.3}$$

Let us denote by F_2 the set-valued map of the right-hand side of differential inclusion (5.3) and let $\mathcal{S}_2$ be the evolutionary system associated with F_2.

PROPOSITION 5.1. *E is viable under the evolutionary system $\mathcal{S}_2$ (5.3).*

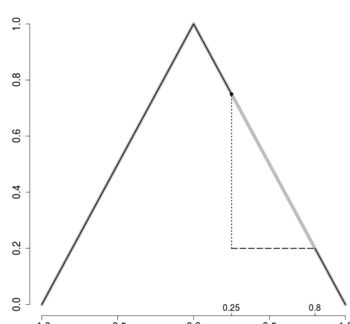

FIG. 5.1. *The graph of the distance function to the boundary of $[-1; 1]$ for $x \in [-1; 1]$ is drawn with a bold gray line. E is drawn with black lines—solid ones for subset E_1, a dotted one for subset E_2, and a dashed one for subset E_3.*

Proof. E is closed. From the tangential characterization of a viability domain,[2] a sufficient condition for E to be viable under $\mathcal{S}_2$ is that $\forall(x,y) \in E$, $F_2(x,y) \cap T_E(x,y) \neq \emptyset$, where $T_E(x,y)$ is the contingent cone of E at (x,y).

From Theorem 4.2, $\mathcal{H}yp(\rho) = \mathrm{Viab}_{\mathcal{S}_1}(\mathcal{H}yp(d))$. If we choose $\gamma(x) := 2kM\ \forall x \in \mathrm{Dom}(F)$ in the definition of $\mathcal{S}_1$ (4.3), we get $\forall(x,y) \in E \cap \mathrm{Int}(\mathcal{H}yp(d)) = E_2 \cup E_3 \cup E_4$, $(F(x) \times [0; 2kM]) \cap T_E(x,y) \neq \emptyset$ from the theorem on p. 146 in [3].

For all $(x,y) \in E_2 \cup E_3 \cup E_4 \setminus \mathrm{Int}_E(E_{24})$, $F_2(x,y) = F(x) \times [0; 2kM]$, so $F_2(x,y) \cap T_E(x,y) \neq \emptyset$.

For all $(x,y) \in \mathrm{Int}_E(E_{24})$, $\exists d > 0$, such that $\mathcal{B}((x,y),d) \cap E \subset \mathrm{Int}_E(E_{24})$. This implies that there exist $\hat{d} > 0$ and $\bar{d} > 0$ such that $\rho(x) \notin [y - \bar{d}, y + \bar{d}]$ when $x \in \mathcal{B}(x, \hat{d})$. Since there exist $z \in F(x)$ and $v \in [0; 2kM]$ such that $(z,v) \in T_E(x,y)$, then there exist $(z_n)_{n\in\mathbb{N}} \in \mathbb{R}^{p^{\mathbb{N}}}$ converging toward z, $(v_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ converging to v, and $(h_n)_{n\in\mathbb{N}} \in [0, +\infty[^{\mathbb{N}}$ converging to 0 such that $\forall n \in \mathbb{N}$, $(x + h_n z_n, y + h_n v_n) \in E$. For n large enough, $x_n + h_n z_n \in \mathcal{B}(x, \hat{d})$ and $h_n \leq \bar{d}/2kM$. Since $(x + h_n z_n, y + h_n v_n) \in E$, $\rho(x_n + h_n z_n) > y + \bar{d}$. $y + h_n 2kM \leq y + \bar{d}$, so $(x + h_n z_n, y + h_n 2kM) \in E$ and $(z, 2kM) \in T_E(x,y)$ and $F_2(x,y) \cap T_E(x,y) \neq \emptyset$.

Let us consider $(x,y) \in E_1 = E \cap \partial\mathcal{H}yp(d)$. $\mathcal{H}yp(\rho) \cap \{(x,\hat{y}) \in \mathbb{R}^p \times [0, +\infty[\mid \hat{y} = y\}$ is a viability domain for F_0 defined by (4.10) from Corollary 4.3. Moreover, $F_0(x,y) \subset F_2(x,y)$, so $F_2(x,y) \cap T_{\mathcal{H}yp(\rho)\cap\{(x,\hat{y})\in\mathbb{R}^p\times[0,+\infty[|\hat{y}=y\}}(x,y) \neq \emptyset$. Then there exist $x' \in F(x)$, two sequences $((d_{xn})_{n\in\mathbb{N}} \in (\mathbb{R}^p)^{\mathbb{N}}, (d_{y_n})_{n\in\mathbb{N}} \in [0, +\infty[^{\mathbb{N}})$ converging to $(x', 0)$, and $(h_n)_{n\in\mathbb{N}}$ converging to 0 such that $\forall n \in \mathbb{N}$,

$$(x + h_n d_{xn}, y + h_n d_{y_n}) \in \mathcal{H}yp(\rho) \cap \{(x,\hat{y}) \in \mathbb{R}^p \times [0, +\infty[| \hat{y} = y\}.$$

Necessarily, $\forall n \in \mathbb{N}$, $d_{y_n} = 0$. Moreover, $\forall n \in \mathbb{N}$, $\rho(x + h_n d_{xn}) \geq \rho(x) = d(x)$ and $\rho(x + h_n d_{xn}) \leq d(x + h_n d_{xn})$. Let us set $v'_n = \frac{\rho(x+h_n d_{xn}) - \rho(x)}{h_n} \leq \frac{d(x+h_n d_{xn}) - d(x)}{h_n} \leq k||d_{xn}||$. Then $\forall n \in \mathbb{N}$, $(x + h_n d_{xn}, y + h_n v'_n) \in E$ with $0 \leq v'_n \leq k||d_{xn}||$. Consequently, there exists a subsequence of $(v'_n)_{n\in\mathbb{N}}$ converging to $V \in [0; k \min_{n\in\mathbb{N}} ||d_{xn}|| \leq kM]$. Finally, $(x', V) \in T_E(x,y) \cap F_2(x,y)$ and $F_2(x,y) \cap T_E(x,y) \neq \emptyset$ $\forall (x,y) \in E_1$. ∎

PROPOSITION 5.3. *If $x \in Viab_{\mathcal{S}}(K)$ and $x(\cdot) \in \mathcal{S}(x) \cap \mathcal{Q}_K(x)$, then the function $t \to \rho(x(t))$ from $[0, +\infty[$ to $[0, +\infty[$ is left-continuous.*

Proof. Let k be the Lipschitz constant of d on $\mathrm{Viab}_{\mathcal{S}}(K)$. Given $t > 0$ and $\epsilon > 0$, $x(\cdot)$ is continuous so $\exists\delta > 0$ such that $\forall h \leq \delta$, $||x(t-h) - x(t)|| \leq \frac{\epsilon}{k}$.

Let us take $h \leq \delta$,

$$\begin{array}{rcl} \rho(x(t-h)) & = & \sup_{(x(t-h),y)\in \mathrm{Viab}_{\mathcal{S}_1}(\mathcal{H}yp(d))} y \\ & = & \sup_{\tilde{x}(\cdot)\in\mathcal{S}(x(t-h))} \inf_{\tau\in[0;+\infty[} d(\tilde{x}(\tau)) \\ & \geq & \sup_{\tilde{x}(\cdot)\in\mathcal{S}(x(t-h)) \mid \forall\tau\in[0;h],\ \tilde{x}(\tau)=x(t-h+\tau)} \inf_{\tau\in[0;+\infty[} d(\tilde{x}(\tau)) \\ & \geq & \min(\min_{\tau\in[t-h;t]} d(x(\tau)), \rho(x(t))) \\ & \geq & \min(d(x(t)) - \epsilon, \rho(x(t))) \\ & \geq & \rho(x(t)) - \epsilon\ . \end{array} \tag{5.4}$$

[2]We recall the theorem on p. 104 in [5].

THEOREM 5.2 (tangential characterization of viability kernels). *Let us assume that F is Marchaud and that $K \subset \mathbb{R}^p$ is a closed subset. The viability kernel $Viab_{\mathcal{S}}(K)$ of a subset K under $\mathcal{S}$ is the largest closed subset D satisfying $D \subset K$ and $\forall x \in D$, $R_D(x(t)) := F(x) \cap T_D(x) \neq \emptyset$. Furthermore, for every $x \in D$, there exists at least one evolution $x(\cdot) \in \mathcal{S}(x)$ viable in D and all evolutions $x(\cdot) \in \mathcal{S}(x)$ viable in D are governed by the differential inclusion $x'(t) \in R_D(x(t))$.*

Then the function $t \to \rho(x(t))$ is lower semicontinuous from the left. Its left continuity follows from the previously established upper semicontinuity of ρ. ☐

Let us consider for $x \in \mathrm{Viab}_{\mathcal{S}}(K)$,

$$\mathcal{S}_K^+(x) := \{x(\cdot) \in \mathcal{S}(x) \cap \mathcal{Q}_K(x) \mid t \to \rho(x(t)) \text{ is non decreasing}\}. \tag{5.5}$$

COROLLARY 5.4. *If $x \in Viab_{\mathcal{S}}(K)$ and $x(\cdot) \in \mathcal{S}_K^+(x)$, then $t \to \rho(x(t))$ is continuous.*

Let us go back to the example of section 2. Since the time derivative of an evolution can be chosen strictly positive in the vicinity of $x = 0.25$ ((2.3) and (2.4)), a viable evolution can increasingly cross the threshold $x = 0.25$. The robustness along this evolution experiences a sudden drop from 0.75 to 0.2; it is then discontinuous but left-continuous since $\rho(0.25) = 0.75$.

Nondecreasing evolutions with ranges in $[-1; 0]$ and nonincreasing evolutions with ranges in $[0; 0.25] \cup [0.8; 1]$ are evolutions along which the robustness is nondecreasing and their robustness is continuous. An evolution which would cross the threshold $x = 0.25$ decreasingly and then experience a sudden positive jump in robustness does not satisfy the differential inclusion describing the dynamics.

COROLLARY 5.5. *If $x \in Viab_{\mathcal{S}}(K)$ and $x(\cdot) \in \mathcal{S}_K^+(x)$, then $t \to \rho(x(t))$ is Lipschitz continuous. Moreover, the Lipschitz constant is not greater than kM with $M := \sup_{x\in K} \sup_{x'\in F(x)} ||x'||$ and k the Lipschitz constant of d on $Viab_{\mathcal{S}}(K)$.*

Proof. Let us consider $t_1 < t_2$.

We first remark that if $\rho(x(t)) < d(x(t))$ $\forall t \in [t_1, t_2]$, then $\rho(x(t_2)) = \rho(x(t_1)) := \rho_0$. In fact, let us define $d_0 = \min_{t\in[t_1,t_2]} d(x(t))$, $d_0 > \rho_0$ since $x(\cdot) \in \mathcal{S}_K^+(x)$ and $d_0 := \rho_0 + \delta$. Since $\rho(x(t_1)) = \rho_0$, $\forall \epsilon > 0, x(t_1)) \notin \mathrm{Viab}_{\mathcal{S}}(K_{\rho_0+\epsilon})$, then $\forall \epsilon > 0$, $\forall y(\cdot) \in \mathcal{S}(x(t_1))$, $\exists T > 0$ such that $y(T) \notin K_{\rho_0+\epsilon}$ that is $d(y(T)) < \rho_0 + \epsilon$. Let us consider $y^*(\cdot) \in \mathcal{S}(x(t_2))$ and the concatenation $y(\cdot)$ of $x(\cdot)$ and $y^*(\cdot)$ defined by $y(t) = x(t_1+t)$ when $t \in [0, t_2-t_1]$ and $y(t) = y^*(t-(t_2-t_1))$ when $t \geq t_2-t_1$. $y(\cdot)$ belongs to $\mathcal{S}(x(t_1))$, so $\forall \epsilon > 0$, $\exists T > 0$ such that $d(y(T)) < \rho_0+\epsilon$. But, $T > t_2-t_1$ when $\epsilon < \delta$, then $d(y^*(T-(t_2-t_1))) < \rho_0+\epsilon$. Consequently, $\forall \epsilon > 0$, $x(t_2) \notin \mathrm{Viab}_{\mathcal{S}}(K_{\rho_0+\epsilon})$ and $\rho(x(t_2)) \leq \rho_0$.

In the general case, let us define t_1^+ and t_2^- by

$$t_1^+ := \sup\{t \geq t_1 \mid \forall t' \in [t_1, t],\ \rho(x(t')) < d(x(t'))\}\ , \tag{5.6}$$

$$t_2^- := \inf\{t \leq t_2 \mid \forall t' \in [t, t_2],\ \rho(x(t')) < d(x(t'))\}\ . \tag{5.7}$$

If $t_2^- < t_1^+$, then $\forall t \in [t_1; t_2]$, $\rho(x(t)) < d(x(t))$ and from the above paragraph, $\rho(x(t_2)) = \rho(x(t_1))$.

If $t_1^+ \leq t_2^-$,

- if $t_1^+ \geq t_1$, $\rho(x(t_1)) = \rho(x(t_1^+)) = d(x(t_1^+))$,
- if $t_1^+ = -\infty$, $\rho(x(t_1)) = d(x(t_1))$,

so, with $T_1 := \max(t_1, t_1^+)$, $\rho(x(t_1)) = d(x(T_1))$, and

- if $t_2^- \leq t_2$, $\rho(x(t_2)) = \rho(x(t_2^-)) = d(x(t_2^-))$,
- if $t_2^- = +\infty$, $\rho(x(t_2)) = d(x(t_2))$,

so, with $T_2 := \min(t_2, t_2^-)$, $\rho(x(t_2)) = d(x(T_2))$.

Then

$$
\begin{aligned}
|\rho(x(t_1)) - \rho(x(t_2))| &= \rho(x(t_2)) - \rho(x(t_1)) \\
&= \rho(x(t_2)) - d(x(T_2)) + d(x(T_2)) - d(x(T_1)) \\
&\quad + d(x(T_1)) - \rho(x(t_1)) \\
&= 0 + d(x(T_2)) - d(x(T_1)) + 0 \\
&\leq kM(t_2 - t_1)\ . \qquad (5.8)
\end{aligned}
$$

□

PROPOSITION 5.6. *If $x \in Viab_{\mathcal{S}}(K)$ and $x(\cdot) \in \mathcal{S}_K^+(x)$, then $(x(\cdot), \rho(x(\cdot))) : [0, +\infty[\to Graph(\rho) = E_1 \cup E_3 \subset E$ is governed by* (5.3)*, i.e., $(x(\cdot), \rho(x(\cdot))) \in \mathcal{S}_2(x, \rho(x))$.*

Proof. $x(\cdot) \in \mathcal{S}(x)$ and from Corollary 5.5, $t \to \rho(x(t))$ is Lipschitz with Lipschitz constant smaller than kM, so $t \to \rho(x(t))$ is absolutely continuous and for almost all $t \geq 0$, the value of the weak derivative $\frac{d\rho(x(t))}{dt} \in [0; kM]$. Consequently, $(x(\cdot), \rho(x(\cdot))) \in \mathcal{S}_2(x, \rho(x))$. □

If we assume that the restriction of the robustness function ρ to $\mathrm{Viab}_{\mathcal{S}}(K)$ is continuous, the converse is true and $\mathrm{Dom}(\mathcal{S}_K^+) = \mathrm{Viab}_{\mathcal{S}}(K)$ as stated in Theorem 5.8 below.

LEMMA 5.7. *If the restriction of the robustness function ρ to $Viab_{\mathcal{S}}(K)$ is continuous, then E and the graph of ρ coincide, $E = Graph(\rho)$.*

Proof. $\mathrm{Graph}(\rho) = E_1 \cup E_3 \subset E$. Let us take $(x, y) \in E$, from the definition of E (5.1), $0 \leq y \leq \rho(x)$, consequently, $x \in \mathrm{Viab}_{\mathcal{S}}(K)$. If $\rho(x) = 0$, $y = 0 = \rho(x)$. In particular, when $x \in \partial\mathrm{Viab}_{\mathcal{S}}(K)$, then $y = \rho(x) = 0$. Otherwise, let us assume that $y < \rho(x)$ and let us take $\epsilon = \rho(x) - y > 0$. Necessarily, $x \in \mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K))$. Since ρ is continuous at x, $\exists\delta > 0$ such that $||x - x'|| \leq \delta$ implies that $||\rho(x) - \rho(x')|| \leq \epsilon/3$. Consequently, $\forall x' \in \mathcal{B}(x, \delta)$, $\rho(x') > y + \epsilon/2$ and $\mathcal{B}(x, \delta) \times [y - \epsilon/2; y + \epsilon/2] \subset Hyp(\rho)$ and $(x, y) \notin E$. □

THEOREM 5.8. *Let us assume that the restriction of the robustness function ρ to $Viab_{\mathcal{S}}(K)$ is continuous; then for any $x \in Viab_{\mathcal{S}}(K)$, there exists $x(\cdot) \in \mathcal{S}_K^+(x)$ and $Dom(\mathcal{S}_K^+) = Viab_{\mathcal{S}}(K)$.*

Proof. Let us consider $x \in \mathrm{Viab}_{\mathcal{S}}(K)$. $(x, \rho(x)) \in E$, so from Theorem 5.1, there exists $(x(\cdot), y(\cdot)) \in \mathcal{S}_2(x, \rho(x))$ viable in E. Since the restriction of ρ to $\mathrm{Viab}_{\mathcal{S}}(K)$ is continuous, from Lemma 5.7, $E = \mathrm{Graph}(\rho)$ and then $\forall t \geq 0$, $\rho(x(t)) = y(t)$. Moreover, $t \to y(t)$ is nondecreasing since $(x(\cdot), y(\cdot))$ is governed by (5.3), so $x(\cdot) \in \mathcal{S}_K^+(x)$. □

Otherwise, if we do not assume that the restriction of the robustness function ρ to $\mathrm{Viab}_{\mathcal{S}}(K)$ is continuous, we get the following properties.

PROPOSITION 5.9. *Given $x \in Viab_{\mathcal{S}}(K)$ with $\rho(x) < d(x)$ (i.e., $(x, \rho(x)) \in E_3$), if $(x(\cdot), y(\cdot)) \in \mathcal{S}_2(x, \rho(x))$ is viable in E, then either $\forall t \geq 0$, $y(t) = \rho(x(t)) < d(x(t))$ (i.e., $(x(t), y(t)) \in E_3$), or $\exists T \geq 0$ such that $y(T) = \rho(x(T)) = d(x(T))$ (i.e., $(x(T), y(T)) \in E_1$) and $\forall t < T$, $y(t) = \rho(x(t)) < d(x(t))$ (i.e., $(x(t), y(t)) \in E_3$).*

Proof. Since $x(0) = x$ and $y(0) = \rho(x) < d(x)$, $(x(0), y(0)) \in E_3$. Let us assume that there exists $T > 0$ such that $(x(T), y(T)) \notin E_3$. Let us take $\tilde{t} := \inf\{t \geq 0 | (x(t), y(t)) \notin E_3\}$, $\tilde{t} \geq 0$, and $(x(\tilde{t}), y(\tilde{t})) \in \overline{E_3} = E_3 \cup (\overline{E_3} \cap E_1) \cup (\overline{E_3} \cap E_{24})$.

- If we assume that $(x(\tilde{t}), y(\tilde{t})) \in (\overline{E_3} \cap E_{24})$, then $\tilde{t} > 0$ and $\rho(x(\tilde{t})) > y(\tilde{t}) = \lim_{t \to \tilde{t}^-} \rho(x(t))$, which contradicts Proposition 5.3.

- If we assume that $(x(\tilde{t}), y(\tilde{t})) \in E_3$, $\rho(x(\tilde{t})) = y(\tilde{t}) < d(x(\tilde{t}))$, so there exists $h > 0$ such that for $0 \leq \tilde{h} \leq h$, $\rho(x(\tilde{t}+\tilde{h})) = \rho(x(\tilde{t})) = y(\tilde{t})$. From (5.3), $\forall t \geq 0$, $y'(t) \geq 0$, so for $0 \leq \tilde{h} \leq h$, $\rho(x(\tilde{t}+\tilde{h})) = y(\tilde{t}+\tilde{h}) < d(x(\tilde{t}+\tilde{h}))$ and $(x(\tilde{t}+\tilde{h}), y(\tilde{t}+\tilde{h})) \in E_3$ which contradicts the definition of $\tilde{t}$.

So, $(x(\tilde{t}), y(\tilde{t})) \in (\overline{E_3} \cap E_1)$. ∎

Proposition 5.10. *Given $x \in Viab_{\mathcal{S}}(K)$ and $(x(\cdot), y(\cdot)) \in \mathcal{S}_2(x, \rho(x))$ viable in E, if there exists $T > 0$ such that $\rho(x(T)) > y(T)$, then there exists $\tilde{T} > 0$, $\tilde{T} \leq T$ such that $(x(\tilde{T}), y(\tilde{T})) \in \overline{E_3} \cap E_{24}$.*

Proof. If $\rho(x(T)) > y(T)$, then $(x(T), y(T)) \in E_{24}$. Let us assume that $(x(T), y(T)) \notin \overline{E_3}$; then $(x(T), y(T)) \in \mathrm{Int}_E(E_{24})$ from (5.2). Let us take $\bar{T} := \inf\{t \geq 0 \mid \forall t' \in [t, T],\ (x(t'), y(t')) \in \mathrm{Int}_E(E_{24})\}$, $\bar{T} \in [0, T]$. From the definition of $\bar{T}$, $(x(\bar{T}), y(\bar{T})) \notin \mathrm{Int}_E(E_{24})$ and $\bar{T} < T$. From Proposition 5.9, $(x(\bar{T}), y(\bar{T})) \notin E_3$.

Let us assume that $(x(\bar{T}), y(\bar{T})) \in E_1$; then $\forall t \in]\tilde{T}, T]$, $(x(t), y(t)) \in \mathrm{Int}_E(E_{24})$. So, from (5.3) $y(T) = y(\bar{T}) + 2kM(T - \bar{T})$, but $(x(\bar{T}), y(\bar{T})) \in E_1$, so $y(\bar{T}) = \rho(x(\bar{T}))$ and $\rho(x(T)) - \rho(x(\bar{T})) \leq kM(T - \bar{T})$ from Proposition 5.5. So $y(T) > \rho(x(T))$, which contradicts $(x(T), y(T)) \in E$. Consequently, $(x(\bar{T}), y(\bar{T})) \notin E_1$.

Finally, $(x(\bar{T}), y(\bar{T})) \in E_{24} \setminus \mathrm{Int}_E(E_{24}) = \overline{E_3} \cap E_{24}$ from (5.2). ∎

Proposition 5.11. *Let us consider $\mathcal{E} \subset E$ defined by*

$$\mathcal{E} := E_1 \cap \overline{E_{24} \cap \overline{E_3}}. \tag{5.9}$$

Given $x \in Viab_{\mathcal{S}}(K)$ and $(x(\cdot), y(\cdot)) \in \mathcal{S}_2(x, \rho(x))$ viable in E, let us take $T :=$ $\sup\{t \geq 0 \mid \forall t' \in [0, t],\ y(t) = \rho(x(t))\}$. If $T < +\infty$, then $(x(T), y(T)) \in \mathcal{E}$.

Proof. From Proposition 5.9, $(x(T), y(T)) \in E_1$. From the definition of T, there exists a sequence $(t_n)_{n\in\mathbb{N}} \to T^+$ such that $\rho(x(t_n)) > y(t_n)$, so $(x(t_n), y(t_n)) \in E_{24}$. Then from Proposition 5.10, $\forall n \in \mathbb{N}$, $\exists t'_n \in]T, t_n]$ such that $(x(t'_n), y(t'_n)) \in E_{24} \cap \overline{E_3}$, so $(x(T), y(T)) \in \overline{E_{24} \cap \overline{E_3}}$ and finally $(x(T), y(T)) \in \mathcal{E}$. ∎

Nevertheless, to obtain results on the existence of increasingly robust evolutions, we have to consider additional conditions:

- The set-valued map F is Lipschitz.
- Let us consider the subsets of $\mathbb{R}^p$,

  $$V^0 := \{x \in \mathrm{Viab}_{\mathcal{S}}(K) \mid \text{the restriction of } \rho \text{ to } \mathrm{Viab}_{\mathcal{S}}(K) \text{ is discontinuous at } x\} \tag{5.10}$$

  and

  $$V^1 := \{x \in V^0 \mid \exists (x_n)_{n\in\mathbb{N}} \in {V^0}^{\mathbb{N}} \mid x = \lim_{n\to\infty} x_n \text{ and } \lim_{n\to\infty} \rho(x_n) < \rho(x)\}. \tag{5.11}$$

  Obviously, if ρ is continuous, then $V^1 = V^0 = \emptyset$.
  Condition (A.1): $V^1 = \emptyset$.
- With $\mathcal{E}$ defined by (5.9), let us take

  $$\mathcal{F} := \{y \geq 0 \mid \exists x \in \mathrm{Viab}_{\mathcal{S}}(K) \mid (x, y) \in \mathcal{E}\} \subset [0; \max_{x\in\mathrm{Viab}_{\mathcal{S}}(K)} \rho(x)]\,. \tag{5.12}$$

  Condition (A.2): $\mathcal{F}$ is a set of isolated points.

Proposition 5.12. *The following statements are equivalent:*

- *$x \in V_0$,*
- *there exists $y < \rho(x)$ such that $(x, y) \in E$.*

Proof. If $x \in V_0$, there exists a sequence $(x_n)_{n\in\mathbb{N}} \in (\mathrm{Viab}_{\mathcal{S}}(K))^{\mathbb{N}}$ such that x_n tends toward x and $\rho(x_n)$ tends toward $y \neq \rho(x)$. For all $n \in \mathbb{N}$, $(x_n, \rho(x_n)) \in E$, which is closed, so (x, y) belongs to E. Let us define $y^* := \frac{\rho(x)+y}{2}$. If $y > \rho(x)$, then $y^* > \rho(x)$, and for n large enough, $x_n \in \mathrm{Viab}_{\mathcal{S}}(K_{y^*})$, which is closed, and then $x \in \mathrm{Viab}_{\mathcal{S}}(K_{y^*})$, which contradicts $\rho(x) < y^*$. So $y < \rho(x)$.

Conversely, if there exists $y < \rho(x)$ such that $(x, y) \in E$, with $y^* := \frac{\rho(x)+y}{2}$, $x \in \partial\mathrm{Viab}_{\mathcal{S}}(K_{y^*})$, so there exists a sequence $(x_n)_{n\in\mathbb{N}} \notin (\mathrm{Viab}_{\mathcal{S}}(K_{y^*}))^{\mathbb{N}}$ which tends toward x. Then there exists a subsequence also denoted by $(x_n)_{n\in\mathbb{N}}$ such that the subsequence $(\rho(x_n))_{n\in\mathbb{N}}$ has a limit which is strictly smaller than $\rho(x)$ since $\rho(x_n) < y^*$ $\forall n \in \mathbb{N}$. ∎

PROPOSITION 5.13. *Let us assume that the set-valued map $F : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$ is Lipschitz. If $x \in V^0$ and $(x(\cdot), y(\cdot)) \in \mathcal{S}_2(x, \rho(x))$ is viable in E, then $\forall t \geq 0$, $x(t) \in V^0$.*

Proof. Let us consider $(x(\cdot), y(\cdot)) \in \mathcal{S}_2(x, \rho(x))$. $x(\cdot) \in \mathcal{S}(x)$ and $\forall t \geq 0$, $y(t) \geq \rho(x)$. Let us assume that $(x(\cdot), y(\cdot))$ is viable in E, then $\forall t \geq 0$, $\rho(x(t)) \geq y(t)$ and then $\rho(x(t)) \geq \rho(x)$ and $x(t) \in \mathrm{Viab}_{\mathcal{S}}(K_{\rho(x)})$.

If $x \in V^0$, from Proposition 5.12, there exists $\bar{y} < \rho(x)$ such that $(x, \bar{y}) \in E$. Let us take $\tilde{y} := \frac{\rho(x)+\bar{y}}{2}$. From (4.9), $x \in \partial\mathrm{Viab}_{\mathcal{S}}(K_{\tilde{y}})$ and $x \in \partial\mathrm{Viab}_{\mathcal{S}}(K_{\rho(x)})$. We remark that since F is Lipschitz, $\mathcal{S}$ is a lower semicontinuous evolutionary system,[3] and then $\mathrm{Viab}_{\mathcal{S}}(K_{\tilde{y}}) \cap \mathrm{Int}(K_{\tilde{y}})$ exhibits the barrier property.[4] Since $x(\cdot)$ is viable in $K_{\rho(x)}$ and $K_{\rho(x)} \subset \mathrm{Int}(K_{\tilde{y}})$, $x(\cdot)$ is also viable in $\mathrm{Int}(K_{\tilde{y}})$ and then $x(\cdot)$ is viable in $\mathrm{Viab}_{\mathcal{S}}(K_{\tilde{y}}) \cap \mathrm{Int}(K_{\tilde{y}})$. Since $\mathrm{Viab}_{\mathcal{S}}(K_{\tilde{y}}) \cap \mathrm{Int}(K_{\tilde{y}})$ exhibits the barrier property, $x(\cdot)$ is actually viable in $\partial\mathrm{Viab}_{\mathcal{S}}(K_{\tilde{y}}) \cap \mathrm{Int}(K_{\tilde{y}})$.

Finally, since $x(\cdot)$ is viable in $\mathrm{Viab}_{\mathcal{S}}(K_{\rho(x)}) \subset \mathrm{Viab}_{\mathcal{S}}(K_{\tilde{y}})$, then $x(\cdot)$ is viable in $\partial\mathrm{Viab}_{\mathcal{S}}(K_{\tilde{y}}) \cap \partial\mathrm{Viab}_{\mathcal{S}}(K_{\rho(x)})$. Since $\rho(x(t)) \geq \rho(x)$ and $\tilde{y} < \rho(x)$, from Proposition 5.12, $\forall t \geq 0$, $x(t) \in V^0$. ∎

THEOREM 5.17. *Let us assume that the set-valued map F is Lipschitz and that conditions* (A.1) *and* (A.2) *hold true; then for any $x \in \mathit{Viab}_{\mathcal{S}}(K)$, there exists $x(\cdot) \in \mathcal{S}^+_K(x)$ and $\mathit{Dom}(\mathcal{S}^+_K) = \mathit{Viab}_{\mathcal{S}}(K)$.*

Proof. Let us consider $x \in \mathrm{Viab}_{\mathcal{S}}(K)$, $(x, \rho(x)) \in E$, which is a viability domain for F_2 from Proposition 5.1, so there exists $(x(\cdot), y(\cdot)) \in S_2(x, \rho(x))$ such that $\forall t \geq 0$, $(x(t), y(t)) \in E$. Let us take $T := \sup\{t \geq 0 \mid \forall t' \in [0, t],\ \rho(x(t')) = y(t')\} \geq 0$, since $y(0) = y = \rho(x) = \rho(x(0))$.

If $T = +\infty$, $(x(\cdot), \rho(x(\cdot))) \in \mathcal{S}_2(x, \rho(x))$ and $x(\cdot) \in \mathcal{S}^+_K(x)$. Otherwise, if $T < +\infty$, from Proposition 5.11, $(x(T), y(T)) \in \mathcal{E}$ and $y(T) \in \mathcal{F}$ defined by (5.12). From the definition of T,

---

[3] See [5, p. 386].

THEOREM 5.14. *Assume that $F : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$ is Lipschitz. Then the associated evolutionary system $\mathcal{S}_F$ is a lower semicontinuous evolutionary system from $\mathbb{R}^p$ into the space of continuous functions supplied with the topology of uniform convergence on compact intervals.*

[4] From [5, p. 408], we have the following.

DEFINITION 5.15 (barrier property). *Let $D \subset \mathbb{R}^p$ be a subset and $\mathcal{S}$ be an evolutionary system. We shall say that D exhibits the barrier property if starting from any $x \in \overset{\circ}{\partial} D := D \cap \overline{\mathbb{R}^p \setminus D}$, all evolutions viable in D on some time interval $[0, T[$ are actually viable in ∂D on $[0, T[$.*

THEOREM 5.16 (barrier property of boundaries of viability kernels). *Assume that $K \subset \mathbb{R}^p$ is closed and that the evolutionary system $\mathcal{S}$ is both upper and lower semicontinuous. Then the intersection $\mathit{Viab}_{\mathcal{S}}(K) \cap \mathrm{Int}(K)$ of the viability kernel of K with the interior of K exhibits the barrier property.*

$$\forall \epsilon > 0,\ \exists h > 0,\ h < \epsilon \mid \rho(x(T+h)) > y(T+h). \tag{5.13}$$

We can then build a sequence $(h_n)_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}$, $0 < h_n$, $(h_n)_{n\in\mathbb{N}}$ is decreasing toward 0 and $\forall n \in \mathbb{N}$, $\rho(x(T+h_n)) > y(T+h_n)$.

Let us consider $x_n := x(T+h_n)$ and $y_n := y(T+h_n)$, $\rho(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ tend toward $y(T)$ with $\forall n \in \mathbb{N}$, $y_n < \rho(x_n)$, so $x_n \in V^0$ defined by (5.10) from Proposition 5.12. Moreover, since $y(T) = \rho(x(T)) \in \mathcal{F}$ which is a set of isolated points from condition (A.2), there exists $\tilde{\rho} > 0$ such that $\forall \tilde{y} \in [\rho(x(T)); \rho(x(T)) + \tilde{\rho}]$, $\tilde{y} \notin \mathcal{F}$, and $\forall \tilde{x} \in \mathrm{Viab}_{\mathcal{S}}(K)$, $(\tilde{x}, \tilde{y}) \notin \mathcal{E}$. Consequently, since $\rho(x_n) \to \rho(x(T))$, $\exists N \in \mathbb{N}$, such that $\forall n \geq N$, $(x_n, \rho(x_n)) \notin \mathcal{E}$.

When $n \geq N$, let us consider $(x_n(\cdot), y_n(\cdot)) \in \mathcal{S}_2(x_n, \rho(x_n))$ viable in E. From Proposition 5.11, while $\rho(x_n(t)) \in [\rho(x(T)); \rho(x(T))+\tilde{\rho}]$, $(x_n(t), y_n(t)) \notin \mathcal{E}$, so $y_n(t) = \rho(x_n(t))$. Moreover, from Proposition 5.13, since $x_n \in V^0$, $\forall t \geq 0$, $x_n(t) \in V^0$. Since F_2 is Marchaud, then $\mathcal{S}_2$ is upper semicompact,[5] so there exists a subsequence of $(x_n(\cdot), y_n(\cdot))_{n\in\mathbb{N}}$ denoted by $(x_n(\cdot), y_n(\cdot))_{n\in\mathbb{N}}$ again which tends toward $(x_1(\cdot), y_1(\cdot)) \in \mathcal{S}_2(x(T), y(T))$ which is also viable in E.

Let us take $T_1 := \sup\{t \geq 0 \mid \forall t' \in [0,t],\ \rho(x_1(t')) = y_1(t')\} \geq 0$. If $T_1 = +\infty$, by concatenating $x(\cdot)_{t\in[0,T]}$ and $x_1(\cdot - T)_{t\in[T,+\infty[}$, we get function $x^*(\cdot) \in \mathcal{S}(x)$ such that $(x^*(\cdot), \rho(x^*(\cdot))) \in \mathcal{S}_2(x, \rho(x))$ and $x^*(\cdot) \in \mathcal{S}_K^+(x)$. If $T_1 < +\infty$, $y_1(T_1) \in \mathcal{F}$.

If we assume that $y_1(T_1) = y(T)$, from the definition of T_1,

$$\forall \epsilon > 0,\ \exists h > 0,\ h < \epsilon \mid \rho(x_1(T_1+h)) > y_1(T_1+h).$$

We can then build a sequence $(h_{1,m})_{m\in\mathbb{N}}$ such that $\forall m \in \mathbb{N}$, $0 < h_{1,m}$, $(h_{1,m})_{m\in\mathbb{N}}$ is decreasing toward 0 and $\forall m \in \mathbb{N}$, $\rho(x_1(T_1+h_{1,m})) > y_1(T_1+h_{1,m})$. Let us take $x_{1,m} := x_1(T_1+h_{1,m})$ and $y_{1,m} := y_1(T_1+h_{1,m})$, $(y_{1,m})_{m\in\mathbb{N}}$ decreases toward $y_1(T_1)$ and $y_{1,m} < \rho(x_{1,m})$.

Since $\lim_{m\to+\infty} \rho(x_{1,m}) = \rho(x_1(T_1)) = y_1(T_1) = y(T)$, there exists $M \in \mathbb{N}$ such that $\forall m \geq M$, $\rho(x_{1,m}) \leq y(T)+\tilde{\rho}$. But $(x_{1,m}, y_{1,m}) = \lim_{n\to\infty}(x_n(T_1+h_{1,m}), y_n(T_1+h_{1,m}))$ so there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, $\forall m \geq M$, $y_n(T_1+h_{1,m}) \leq y(T)+\tilde{\rho}$, and then $y_n(T_1+h_{1,m}) = \rho(x_n(T_1+h_{1,m}))$ from Proposition 5.11.

Hence $x_{1,m} = \lim_{n\to\infty} x_n(T_1+h_{1,m})$ with $x_n(T_1+h_{1,m}) \in V^0$ and $\lim_{n\to\infty} \rho(x_n(T_1+h_{1,m})) = y_{1,m} < \rho(x_{1,m})$ so $x_{1,m} \in V_1$, which contradicts condition (A.1) and necessarily $y_1(T_1) > y(T)$.

At stage p, let us set $T_p := \sup\{t \geq 0 \mid \forall t' \in [0,t],\ \rho(x_p(t')) = y_p(t')\}$; if $T_p < +\infty$, we get the strictly increasing sequence $(y_i(T_i))_{i\in\{1,\dots,p\}}$ of elements of $\mathcal{F}$. Since $\mathcal{F}$ is a discrete subset of $[0,+\infty[$, it is a finite or countable set. If there exists $p \in \mathbb{N}$ such that $T_p = +\infty$, the function $x^*(\cdot) \in \mathcal{S}(x)$ defined by the concatenation of $x(\cdot)_{t\in[0,T]}$, $x_1(\cdot - T)_{t\in[T,T+T_1]}$, $\dots$, $x_i(\cdot - (T + \sum_{j=1\dots i-1} T_j)_{t\in[T+\sum_{j=1\dots i-1} T_j, T+\sum_{j=1\dots i-1} T_j + T_i]}$, $\dots$, $x_p(\cdot - (T + \sum_{j=1\dots p-1} T_j)_{t\in[T+\sum_{j=1\dots p-1} T_j, +\infty[}$ is such that $(x^*(\cdot), \rho(x^*(\cdot))) \in$

[5] We recall the definitions and results from [5, p. 384].

Definition 5.18. *Let $\mathcal{S} : \mathbb{R}^p \rightsquigarrow \mathcal{C}([0,\infty[;\mathbb{R}^p)$ be an evolutionary system where both the state space $\mathbb{R}^p$ and the evolutionary space $\mathcal{C}([0,\infty[;\mathbb{R}^p)$ are topological spaces. The evolutionary system is said to be upper semicompact at x if for every sequence $x_n \in \mathbb{R}^p$ converging to x and for every sequence $x_n(\cdot) \in \mathcal{S}(x_n)$, there exists a subsequence $x_{n_p}(\cdot)$ converging to some $x(\cdot) \in \mathcal{S}(x)$. It is said to be an upper semicompact if it is upper semicompact at every point $x \in \mathbb{R}^p$ where $\mathcal{S}(x)$ is not empty.*

Theorem 5.19. *If $F : \mathbb{R}^p \rightsquigarrow \mathbb{R}^p$ is Marchaud, the associated evolutionary system $\mathcal{S}$ is upper semicompact.*

$\mathcal{S}_2(x,\rho(x))$. Otherwise, the function $x^*(\cdot) \in \mathcal{S}(x)$ defined by the concatenation of $x(\cdot)_{t\in[0,T]}, \; \ldots, \; x_i(\cdot - (T+\sum_{j=1\ldots i-1} T_j)_{t\in[T+\sum_{j=1\ldots i-1} T_j, T+\sum_{j=1\ldots i-1} T_j+T_i]}, \; \ldots$ for $i \in \mathbb{N}$ is such that $(x^*(\cdot),\rho(x^*(\cdot))) \in \mathcal{S}_2(x,\rho(x))$ and $x^*(\cdot) \in \mathcal{S}_K^+(x)$. $\square$

6. Regulating increasingly robust evolutions. In this section, we consider particular differential inclusions provided by controlled dynamical systems (U,f) (see [3, 5]) defined by

$$\left\{\begin{array}{rcl} x'(t) & = & f(x(t),u(t)), \\ u(t) & \in & U(x(t)), \end{array}\right. \tag{6.1}$$

where the evolution of the state $x(\cdot)$ ranges over a finite dimensional vector space $\mathbb{R}^p$ and the evolution of the control $u(\cdot)$ ranges over a finite dimensional vector space $\mathbb{R}^q$. The set-valued map $U : \mathbb{R}^p \rightsquigarrow \mathbb{R}^q$ describes the state-dependent constraints on the controls ($U(x) \subset \mathbb{R}^q$ is the set of admissible controls when the state of the system is x) and f is a function from $\mathrm{Graph}(U)$ to $\mathbb{R}^p$.

From (U,f), we can define the set-valued map F which associates with any x the subset $F(x) := \{f(x,u)\}_{u\in U(x)}$ of velocities parameterized by $u \in U(x)$. The associated evolutionary system $\mathcal{S}$ maps any initial state x to the set $\mathcal{S}(x)$ of evolutions $x(\cdot)$ starting from x and governed by

$$x'(t) = f(x(t),u(t)) \text{ where } u(t) \in U(x(t)) \tag{6.2}$$

or, equivalently, to differential inclusion $x'(t) \in F(x(t))$. When the set-valued map F is Marchaud, we shall say that the controlled dynamical system (U,f) is Marchaud.

As in the previous section, we assume that the conditions of Theorem 4.2 are satisfied; particularly the controlled dynamical system (U,f) is Marchaud, K is compact, $\mathrm{Viab}_{\mathcal{S}}(K) \neq \emptyset$, $M = \sup_{x\in K}\sup_{x'\in F(x)} ||x'|| < +\infty$, and k is the Lipschitz constant on $\mathrm{Viab}_{\mathcal{S}}(K)$ of the extended function associated with the map of nominal shocks $D \in \mathcal{M}$.

Lemma 6.1. *Let us assume that* $Dom(\mathcal{S}_K^+) = Viab_{\mathcal{S}}(K)$. *If* $x(\cdot) \in \mathcal{S}_K^+(x)$, *then the evolution* $(x(\cdot),\rho(x(\cdot)))$ *is governed by*

$$\left\{\begin{array}{rcl} x'(t) & = & f(x(t),u(t)), \\ y'(t) & = & v(t) \end{array}\right. \tag{6.3}$$

with $(u(t),v(t)) \in R_{ob}(x(t),y(t))$, *where* R_{ob} *is the set-valued map defined* $\forall (x,y) \in Graph(\rho)$ *by*

$$R_{ob}(x,y) \;:=\; \{(u,v) \in U(x) \times [0;2kM] \mid (f(x,u),v) \in T_{Graph(\rho)}(x,y)\}. \tag{6.4}$$

Proof. Let us consider $x \in \mathrm{Viab}_{\mathcal{S}}(K)$. If $x(\cdot) \in \mathcal{S}_K^+(x)$, $(x(\cdot),\rho(x(\cdot))) \in \mathcal{S}_2(x,\rho(x))$ from Proposition 5.6. But, $\forall t \geq 0$, $(x(t),\rho(x(t))) \in \mathrm{Graph}(\rho)$, so for almost all $t \geq 0$, $(x'(t),\rho'(x(t))) \in T_{\mathrm{Graph}(\rho)}(x(t),\rho(x(t))) \cap F_2(x(t),\rho(x(t)))$. Then since in the particular case of controlled dynamical system $F_2(x(t),\rho(x(t))) = \{f(x,u)\}_{u\in U(x)} \times [0;2kM]$, $(x(\cdot),\rho(x(\cdot)))$ is governed by (6.3) and (6.4).

The following theorem allows us to determine the maximal robustness value which is reachable by increasingly robust evolutions and to govern increasingly robust evolutions which reach a given level of robustness according to a given time horizon.

Let us consider for $(x, y, z, \tau) \in \text{Graph}(\rho) \times \mathbb{R} \times \mathbb{R}$ the auxiliary controlled dynamical system:

$$\text{(6.5)} \qquad \begin{cases} x'(t) &= f(x(t), u(t)), \\ y'(t) &= v(t), \\ z'(t) &= -v(t), \\ \tau'(t) &= -1, \end{cases}$$

where $(u(t), v(t)) \in R_{ob}(x, y)$ with $R_{ob}(x, y)$ defined by (6.4). Let $\mathcal{S}_3$ be the evolutionary system associated with (6.5). Let us consider the constraint set:

$$\text{(6.6)} \qquad K^c := \{(x, y, z, \tau) \in \text{Graph}(\rho) \times \mathbb{R} \times \mathbb{R} |\ z \geq 0 \text{ and } \tau \geq 0\}$$

and the target

$$\text{(6.7)} \qquad C^c := \{(x, y, z, \tau) \in \text{Graph}(\rho) \times \mathbb{R} \times \mathbb{R} |\ z = 0 \text{ and } \tau \geq 0\}.$$

Theorem 6.2. *If $Dom(\mathcal{S}_K^+) = Viab_{\mathcal{S}}(K)$, then*

$$\text{(6.8)} \qquad z_{sup}(x) := \sup_{(x,y,z,\tau) \in Viab_{\mathcal{S}_3}(K^c, C^c)} z = \sup_{x(\cdot) \in \mathcal{S}_K^+(x)} \sup_{t \geq 0} \rho(x(t)) - \rho(x),$$

where $Viab_{\mathcal{S}_3}(K^c, C^c)$ is the viability kernel with target.[6]

Moreover,

$$\text{(6.9)} \qquad T_{inf}(x, z) := \inf_{(x,y,z,\tau) \in Viab_{\mathcal{S}_3}(K^c, C^c)} \tau = \inf_{x(\cdot) \in \mathcal{S}_K^+(x)} \inf\{t \geq 0 \mid \rho(x(t)) - \rho(x) \geq z\}$$

is the minimal time necessary to increase the robustness value from $\rho(x)$ to $\rho(x) + z$ along an increasingly robust viable evolution starting at x.

Proof. Let us take $(x, y, z, \tau) \in \text{Graph}(\rho) \times \mathbb{R} \times \mathbb{R}$. Since $\text{Dom}(\mathcal{S}_K^+) = \text{Viab}_{\mathcal{S}}(K)$, there exists $x(\cdot) \in \mathcal{S}_K^+(x)$. Let us consider $y(\cdot) := \rho(x(\cdot))$, $z(\cdot) := \rho(x) - \rho(x(\cdot)) + z$ and $\tau(\cdot) := \tau - \cdot$; then $(x(0), y(0), z(0), \tau(0)) = (x, y, z, \tau)$ and from Lemma 6.1 $(x(\cdot), y(\cdot), z(\cdot), \tau(\cdot))$ is governed by (6.5) and $\text{Dom}(\mathcal{S}_3) = \text{Graph}(\rho) \times \mathbb{R} \times \mathbb{R}$.

We remark that the dynamics of x, y, z, and τ do not depend on z and τ, so, if $(x, y, z, \tau) \in \text{Viab}_{\mathcal{S}_3}(K^c, C^c)$, then $\forall z'$ such that $0 \leq z' \leq z$, $(x, y, z', \tau) \in \text{Viab}_{\mathcal{S}_3}(K^c, C^c)$ and $\forall \tau'$ such that $\tau' \geq \tau$, $(x, y, z, \tau') \in \text{Viab}_{\mathcal{S}_3}(K^c, C^c)$.

Let us take $x \in \text{Viab}_{\mathcal{S}}(K)$ and $z < z_{sup}(x)$; then there exists T such that $(x, \rho(x), z, T) \in \text{Viab}_{\mathcal{S}_3}(K^c, C^c)$. Consequently, there exist $(x(\cdot), y(\cdot), z(\cdot), \tau(\cdot))$ governed by (6.5) such that $(x(0), y(0), z(0), \tau(0)) = (x, \rho(x), z, T)$ and $\hat{T} \in [0, T]$ such that $(x(t), y(t), z(t), \tau(t)) \in K^c\ \forall t \in [0; \hat{T}]$ and $z(\hat{T}) = 0$. That is, $\rho(x(t)) = \rho(x) + z - z(t)\ \forall t \in [0; \hat{T}]$ and consequently, $\rho(x(\hat{T})) = \rho(x) + z$. Let us complete $x(\cdot)$ for $t \geq T$ by $\tilde{x}(\cdot - T)$ with $\tilde{x}(\cdot) \in \mathcal{S}_K^+(x(T))$; the concatenation is again denoted by $x(\cdot)$. Then $x(\cdot) \in \mathcal{S}_K^+(x)$, $\sup_{t \geq 0} \rho(x(t)) - \rho(x) \geq z$ and $\sup_{x(\cdot) \in \mathcal{S}_K^+(x)} \sup_{t \geq 0} \rho(x(t)) - \rho(x) \geq z_{sup}(x)$.

[6] We recall the definition from [5, p. 86].

Definition 6.3. *Let $\mathcal{S} : \mathbb{R}^p \rightsquigarrow \mathcal{C}([0, \infty[; \mathbb{R}^p)$ be an evolutionary system. Given $C \subset K \subset \mathbb{R}^p$, the subset $Viab_{\mathcal{S}}(K, C)$ of initial states $x_0 \in K$ such that at least one evolution in $\mathcal{S}(x_0)$ starting at x_0 is viable in K $\forall t \geq 0$ or viable in K until it reaches C in finite time is called the viability kernel of K with target C under $\mathcal{S}$.*

Let us take $x \in \mathrm{Viab}_{\mathcal{S}}(K)$ and $x(\cdot) \in \mathcal{S}_K^+(x)$. For all $t \geq 0$, $(x(0), \rho(x(0)), \rho(x(t)) - \rho(x(0)), t) \in \mathrm{Viab}_{\mathcal{S}_3}(K^c, C^c)$. Actually, $\forall t' \in [0;t]$, $(x(t'), \rho(x(t'))) \in \mathrm{Graph}(\rho)$, $z(t') = \rho(x(t)) - \rho(x(t')) \geq 0$, and $\tau(t') = t - t' \geq 0$, and $z(t) = 0$. Finally, $(x, \rho(x), \rho(x(t)) - \rho(x), t) \in \mathrm{Viab}_{\mathcal{S}_3}(K^c, C^c)$, $\sup_{t \geq 0} \rho(x(t)) - \rho(x) \leq z_{sup}(x)$ and $\sup_{x(\cdot) \in \mathcal{S}_K^+(x)} \sup_{t \geq 0} \rho(x(t)) - \rho(x) \leq z_{sup}(x)$.

We use the same reasoning to prove the second equality. Let us take $x \in \mathrm{Viab}_{\mathcal{S}}(K)$, $z \leq z_{sup}(x)$ and $T > T_{inf}(x,z)$; then $(x, \rho(x), z, T) \in \mathrm{Viab}_{\mathcal{S}_3}(K^c, C^c)$. Consequently, there exist $(x(\cdot), y(\cdot), z(\cdot), \tau(\cdot))$ governed by (6.5) such that $(x(0), y(0), z(0), \tau(0)) = (x, \rho(x), z, T)$, and $\hat{T} \in [0, T]$ such that $(x(t), y(t), z(t), \tau(t)) \in K^c$ $\forall t \in [0; \hat{T}]$ and $z(\hat{T}) = 0$. That is $\rho(x(t)) = \rho(x) + z - z(t)$ $\forall t \in [0; \hat{T}]$ and consequently, $\rho(x(\hat{T})) = \rho(x) + z$. Let us complete $x(\cdot)$ for $t \geq T$ by $\tilde{x}(\cdot - T)$ with $\tilde{x}(\cdot) \in \mathcal{S}_K^+(x(T))$; the concatenation is again denoted by $x(\cdot)$. Then $x(\cdot) \in \mathcal{S}_K^+(x)$, $\inf\{t \geq 0 \mid \rho(x(t)) - \rho(x) \geq z\} \leq T$ and $\inf_{x(\cdot) \in \mathcal{S}_K^+(x)} \inf\{t \geq 0 \mid \rho(x(t)) - \rho(x) \geq z\} \leq T_{inf}(x,z)$.

Let us take $x \in \mathrm{Viab}_{\mathcal{S}}(K)$ and $x(\cdot) \in \mathcal{S}_K^+(x)$. For all $t \geq 0$, $(x(0), \rho(x(0)), \rho(x(t)) - \rho(x(0)), t) \in \mathrm{Viab}_{\mathcal{S}_3}(K^c, C^c)$. Actually, $\forall t' \in [0;t]$, $(x(t'), \rho(x(t'))) \in \mathrm{Graph}(\rho)$, $z(t') = \rho(x(t)) - \rho(x(t')) \geq 0$, and $\tau(t') = t - t' \geq 0$, and $z(t) = 0$. Finally, $(x, \rho(x), \rho(x(t)) - \rho(x), t) \in \mathrm{Viab}_{\mathcal{S}_3}(K^c, C^c)$, so if there exists $t \geq 0$ such that $\rho(x(t)) - \rho(x) = z$, then $(x, \rho(x), z, t) \in \mathrm{Viab}_{\mathcal{S}_3}(K^c, C^c)$, $\inf\{t \geq 0 \mid \rho(x(t)) - \rho(x) \geq z\} \geq T_{inf}(x,z)$ and $\inf_{x(\cdot) \in \mathcal{S}_K^+(x)} \inf\{t \geq 0 \mid \rho(x(t)) - \rho(x) \geq z\} \geq T_{inf}(x,z)$. ∎

7. Conclusion. When studying the compatibility between a controlled dynamical system and a constraint set, the viability kernel discriminates the states from which there exists at least one evolution which remains in this constraint set. Moreover, the viability kernel is invariant if the sets of admissible controls are reduced to the regulation map built upon the viability kernel. From a control perspective, the question arises to build single-valued control maps governing viable evolutions, which are usually called feedbacks in viability theory framework [5]. One approach is to build a particular viable feedback as a retroaction governing optimal evolutions which optimize an intertemporal criterion (involving integral or nonintegral functionals, with finite or infinite time horizon). Another approach that does not require the definition of an optimization criterion is to find constructive selections of the regulation map. For example, the minimal selection which considers the control of the regulation map with minimal norm can provide viable evolutions [10] under appropriate assumptions [14]. However, when the manager considers the possibility of the occurrence of unexpected disturbances (perturbations which are not included in the dynamics used to derive the viability kernel), a valuable viable feedback law may be the one that would ensure the viability against the largest set of unexpected disturbances.

In this article, we have considered unexpected disturbances described by a single jump in the state space that may occur at any time in the future. Regarding this type of disturbance, we have defined the robustness function which associates each point of the viability kernel with the maximal value of the jump size the system can support now and in the future without leaving the viability kernel. Then we have shown how to evaluate the robustness function since its hypograph can be regarded as a viability kernel of an auxiliary system. Hence the robustness function can be computed thanks to algorithms used for viability kernel approximation (Theorem 4.2). Finally, we have shown how to build a constructive selection of the regulation map that governs increasingly robust evolutions (Lemma 6.1) and how to govern particular increasingly robust evolutions as solutions of a particular target reachability problem using a minimal time criterion (Theorem 6.2). The interest is that along such evolutions the size of the disturbance the system can support without leaving the viability kernel is nondecreasing with time.

REFERENCES

[1] I. Alvarez and S. Martin, *Geometric robustness of viability kernels and resilience basins*, in Viability and Resilience of Complex Systems: Understanding Complex Systems, G. Deffuant and N. Gilbert, eds., Springer, Berlin, 2011, pp. 193–218.

[2] I. Alvarez, S. Martin, and S. Mesmoudi, *Describing the result of a classifier to the end-user: Geometric-based sensitivity*, in 19th European Conference on Artificial Intelligence, Front. Arti. Intell. Appl. 215, H. Coelho, R. Studer, and M. Wooldridge, eds., IOS Press, Amsterdam, 2010, pp. 835–840, http://dx.doi.org/10.3233/978-1-60750-606-5-835.

[3] J.-P. Aubin, *Viability Theory*, Birkhäuser, Basel, 1991.

[4] J.-P. Aubin, *Tychastic viability*, Acta Biotheor., 61 (2013), pp. 329–340, https://doi.org/10.1007/s10441-013-9194-4.

[5] J.-P. Aubin, A. Bayen, and P. Saint-Pierre, *Viability Theory: New Directions*, Springer, New York, 2011.

[6] J.-P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin, 1983.

[7] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Basel, 1990.

[8] J.-P. Aubin and H. Frankowska, *The viability kernel algorithm for computing value functions of infinite horizon optimal control problems*, J. Math. Anal. Appl., 201 (1996), pp. 555–576.

[9] J.-P. Aubin and G. D. Prato, *Stochastic Nagumo's viability theorem*, Stoch. Anal. Appl., 13 (1995), pp. 1–11.

[10] C. Bernard and S. Martin, *Comparing the sustainability of different action policy possibilities: Application to the issue of both household survival and forest preservation in the corridor of Fianarantsoa*, Math. Biosci., 245 (2013), pp. 322–330.

[11] F. Blanchini, *Set invariance in control*, Automatica, 35 (1999), pp. 1747–1767, http://www.sciencedirect.com/science/article/pii/S0005109899001132.

[12] P. Cardaliaguet, *A differential game with two players and one target*, SIAM J. Control Optim., 34 (1996), pp. 1441–1460, http://dx.doi.org/10.1137/S036301299427223X.

[13] P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre, *Pursuit differential games with state constraints*, SIAM J. Control Optim., 39 (2001), pp. 1615–1632.

[14] M. Falcone and P. Saint-Pierre, *Slow and quasi-slow solutions of differential inclusions*, Nonlinear Anal., 11 (1987), pp. 367–377.

[15] A. F. Filippov, *Classical solutions of differential equations with multivalued right-hand side*, SIAM J. Control Optim., 5 (1967), pp. 609–621.

[16] E. C. Kerrigan and J. M. Maciejowski, *Invariant sets for constrained nonlinear discrete-time systems with application to feasibility in model predictive control*, in Proceedings of the 39th Conference on Decision and Control, IEEE, 2000.

[17] J. N. Maidens, S. Kaynama, I. M. Mitchell, M. M. Oishi, and G. A. Dumont, *Lagrangian methods for approximating the viability kernel in high-dimensional systems*, Automatica, 49 (2013), pp. 2017–2029, http://www.sciencedirect.com/science/article/pii/S0005109813001891.

[18] S. Martin, *The cost of restoration as a way of defining resilience: a viability approach applied to a model of lake eutrophication*, Ecol. Soc., 9 (2004).

[19] I. M. Mitchell, A. M. Bayen, and C. J. Tomlin, *A time-dependant Hamilton-Jacobi formulation of reachable sets for continuous dynamics games*, IEEE Trans. Automat. Control, 50 (2005), pp. 947 – 957.

[20] G. D. Prato and H. Frankowska, *Invariance of stochastic control systems with deterministic arguments*, J. Differential Equations, 200 (2004), pp. 18–52.

[21] M. Quincampoix, *Differential inclusions and target problems*, SIAM J. Control Optim., 30 (1992), pp. 324–335.

[22] P. Saint-Pierre, *Approximation of viability kernel*, Appl. Math. Optim., 29 (1994), pp. 187–209.

[23] C. J. Tomlin, J. Lygeros, and S. S. Sastry, *A game theoretic approach to controller design for hybrid systems*, Proc. IEEE, 88 (2000), pp. 949–970, https://doi.org/10.1109/5.871303.